动力系统分析

第二章 动力系统基本概念

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2.1 动力系统的定义

定义（动力系统）： M是C^r流形，φ是C^r（或连续）映射，$\varphi: \mathbb{R}\times M \to M$，若$\varphi$满足：

(a) $\varphi(0,x)=x \quad \forall x\in M$ （恒等）

(b) $\varphi(s+t,x)=\varphi(s,\varphi(t,x)) \quad \forall s,t\in\mathbb{R},\ \forall x\in M$

则φ是M上的C^r（或连续）流，也称作动力系统。

若φ是M上的连续流，则对于任意给定的时间$t\in\mathbb{R}$，$\varphi^t\equiv\varphi(t,x)$都定义了一个连续的映射$\varphi^t: M\to M$，且：$\varphi^0=\varphi^{-t+t}=id$(恒等映射)，$\varphi^{t+s}=\varphi^t\circ\varphi^s=\varphi^s\circ\varphi^t$ $(\forall t,s\in\mathbb{R})$

第一点，$t=0$对应的是恒等映射$\varphi(0,x)=x$
第二点，先做t的映射再做s映射等价于做了$t+s$的映射$\varphi(s+t,x)=\varphi(s,\varphi(t,x))$

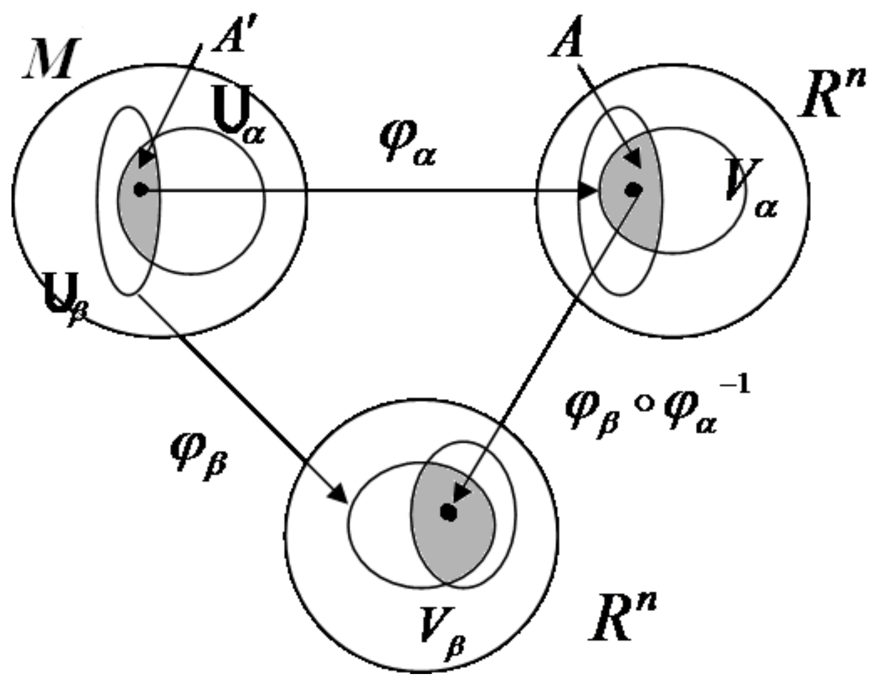
这两点构成了因果律

定义（流形）： 集合M称作n维流形(manifold)，如果满足：

(a) U_α是M的开集，$\bigcup\limits_\alpha U_\alpha = M$，且存在一对一的映射，$\varphi_\alpha$：$U_\alpha \to V_\alpha \subset \mathbb{R}^n$

(b) 若$U_\alpha \bigcap U_\beta \neq \emptyset$，则$\varphi_\beta \circ \varphi_\alpha^{-1}$：$\varphi_\alpha(U_\alpha \bigcap U_\beta) \to \varphi_\beta(U_\alpha \bigcap U_\beta)$是$C^r$ 的；

(c) 若$x \in U_\alpha$，$\tilde{x} \in U_\beta$且$x \neq \tilde{x}$，则存在$\mathbb{R}^n$ 中的开集$W_c \subset V_c \subset \mathbb{R}^n$ $\tilde{W} \subset V_\beta \subset \mathbb{R}^n$，$\varphi_\alpha(x) \in V_\alpha$，$\varphi_\beta(\tilde{x}) \in V_\beta$使得$\varphi_\alpha^{-1}(W) \bigcap \varphi_\beta^{-1}(\tilde{W}) = \emptyset$.



流形就是在一个集合上加上拓扑(a)和微分结构(b)后形成的一个在局部和欧氏空间$\mathbb{R}^n$上一样(亦即可以建立同胚映射)的数学对象，但其整体上未必和$\mathbb{R}^n$相同

映射： 设集合X,Y为两个非空集合，一个从X到Y的映射，记作$f:X\to Y$,这个映射是一个法则，它给X 的每一个元素定义了唯一的Y中的元素$x\mapsto y, x\in X, y\in Y$

连续： 两个定义了开集的集合X与Y之间的映射$f:X\to Y$称为连续的，如果Y中的开集在这个映射的逆映射下对应于X中的开集。

定义（轨道）： 对一个动力系统$\varphi:\mathbb{R}\times M\to M$，任意给定一点$x\in M$，则点集$Orb_{\varphi(x)}=\{\varphi(t,x);\ t\in\mathbb{R}\}\subset M$称作流$\varphi$过点$x$的轨道。

若φ是M上的连续流，则对于任意给定的时间$t \in \mathbb{R}$, $\varphi^t \equiv \varphi(t,x)$都定义了一个连续的映射$\varphi^t : M \to M$，且：$\varphi^0 = \varphi^{-t+t} = id$(恒等映射), $\varphi^{t+s} = \varphi^t \circ \varphi^s = \varphi^s \circ \varphi^t \ (\forall t, s \in \mathbb{R})$

若φ^t关于时间连续可微，则该动力系统和一个微分系统等价：

令 $$f(x) = \frac{d}{dt}\varphi^t(x)\Big|_{t=0} = \lim_{\Delta t \to 0} \frac{\varphi^{\Delta t}(x) - \varphi^0(x)}{\Delta t}$$

则
$$\begin{aligned}
\frac{d}{dt}\varphi^t(x) &= \lim_{\Delta t \to 0} \frac{\varphi^{t+\Delta t}(x) - \varphi^t(x)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\varphi^{\Delta t}(\varphi^t(x)) - \varphi^0(\varphi^t(x))}{\Delta t} \\
&= \lim_{\Delta \alpha \to 0} \frac{\varphi^{\Delta \alpha}(\varphi^t(x)) - \varphi^0(\varphi^t(x))}{\Delta \alpha} \\
&= \frac{d}{d\alpha}\varphi^\alpha(\varphi^t(x))\,|_{\alpha=0} \\
&= f(\varphi^t(x))
\end{aligned}$$

$$\frac{d}{dt}\varphi^t(x) = f(\varphi^t(x))$$

令$y = \varphi^t(x)$，则，$\dfrac{dy}{dt} = f(y)$

对连续流抽样：$\dots\varphi^{-n\tau},\varphi^{-(n-1)\tau}\dots\varphi^{-2\tau},\varphi^{-\tau},id,\varphi^{\tau},\varphi^{2\tau}\dots\varphi^{n\tau}\dots$ 构成离散动力系统。相应地，如果同胚映射f满足：

a) $f^0 = id;$

b) $f^{k+l} = f^k \circ f^l \quad \forall k,l \in \mathbb{Z}$

则构成一离散动力系统。

一般地，任意同胚映射并不一定是某个流在时刻τ的映射

流可以通过采样自然地生成一个离散动力系统，但离散动力系统并不一定能直接嵌入某个流作为时刻τ的映射。

连续动力系统对应于微分方程，而离散动力系统对应于差分方程。

$$\dot{x} = f(x, t)$$

$$x_{t+1} = f(x_t, x_{t-1}, \ldots)$$

通常只考虑一阶的微分方程和差分方程 $x^{(n)} = f(x^{(n-1)}, x^{(n-2)}, \ldots, \dot{x}, t)$ $x^{(n)} = \dfrac{d^n x}{dt^n}$

令 $x^{(n-1)} = y_n \ \ldots\ , x^{(1)} = y$

$$\begin{cases} \dfrac{dy_n}{dt} = f(y_n, y_{n-1}, \ldots, t) \\ \dfrac{dy_{n-1}}{dt} = y_n \\ \ldots \\ \dfrac{dx}{dt} = y \end{cases}$$

例如： $\ddot{x} + ax = 0$，令 $\dot{x} = y \Rightarrow \begin{cases} \dfrac{dy}{dt} = -ax \\ \dfrac{dx}{dt} = y \end{cases}$

差分系统也可能高阶差分的系统，如：

$$x_t = f(x_{t-1}, x_{t-2}, \ldots, x_{t-n})$$

则令，$x_{t-1} = y_{1,t}$，$x_{t-2} = y_{2,t}$，$\ldots, x_{t-n} = y_{n,t}$，于是

$$\begin{cases} y_{1,t+1} = x_t = f(y_{1,t}, y_{2,t}, \ldots, y_{n,t}) \\ y_{2,t+1} = y_{1,t} \\ y_{3,t+1} = y_{2,t} \\ \cdots \\ y_{n,t+1} = y_{n-1,t} \end{cases}$$

Example

eg1

$e^{At}x$是一个线性流，其中A是$n\times n$矩阵，$x\in\mathbb{R}^n$，$t\in\mathbb{R}$，

对应于微分动力系统：$\dot{x}=Ax$

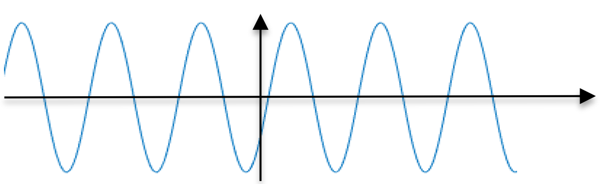
eg2　无阻尼自由振动

$$\frac{d^2x}{dt^2}+\omega_0^2x=0$$

$$x(t)=c_1\cos\omega_0 t+c_2\sin\omega_0 t$$

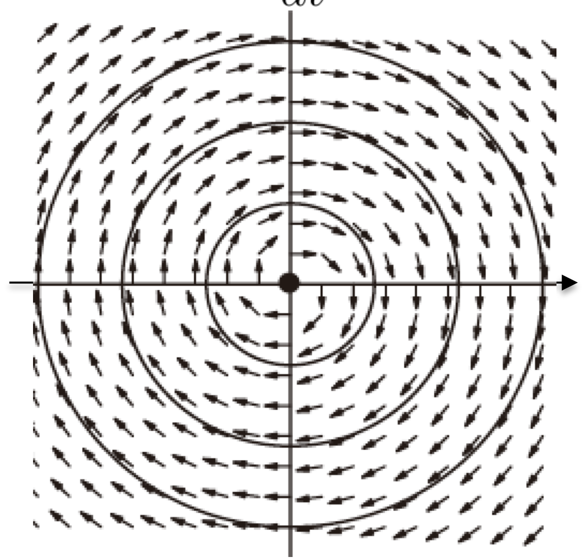
$$A=\sqrt{c_1^2+c_2^2},\ \theta=\arctan\frac{c_1}{c_2}$$

$$\begin{aligned}x(t)&=\sqrt{c_1^2+c_2^2}\left[\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\omega_0 t+\frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\omega_0 t\right]\\&=A(\sin\theta\cos\omega_0 t+\cos\theta\sin\omega_0 t)\\&=A\sin(\omega_0 t+\theta)\end{aligned}$$



let $y=\frac{1}{\omega}\frac{dx}{dt}$

$$\frac{dx}{dt}=\omega y\qquad\frac{dy}{dt}=-\omega x$$



微分方程中的时间 t

$\dot{x} = f(x,t)$　　右边显含时间，称作非自治微分方程

$\frac{d}{dt}\varphi^t(x) = f(\varphi^t(x))$ 右边不显含时间，称作自治微分方程

1) 时间平移：若$x = \varphi(t)$是$\dot{x} = f(x)$的解，则$\varphi(t - t_0)$也是方程的解。理由是，令$t - t_0 = \tau$，则$\frac{d}{d\tau} = \frac{d}{dt}$，即

$$\frac{d}{d\tau}x = f(x, t_0 + \tau) = f(x)$$

2) 时间可加性：

$$\varphi_{t_1+t_2}(x_0) = \varphi_{t_1}(\varphi_{t_2}(x_0)) = \varphi_{t_2}(\varphi_{t_1}(x_0))$$

由动力系统的定义可知连续可微的动力系统对应一个自治微分方程，非自治微分方程有时候并不满足动力系统的定义。自治微分方程反映的系统自身的演化机制，不考虑外界的驱动和控制。非自治微风方程可看作是有外界驱动和控制的，有时候非自治系统可以嵌入到一个高维的自治系统中。

$$\dot{x} = y$$
$$\dot{y} = -x + \cos \omega t$$

等价于

$$\dot{x} = y$$
$$\dot{y} = -x + z_1$$
$$\dot{z}_1 = -\omega z_2$$
$$\dot{z}_2 = \omega z_1$$

时间延迟系统: $\dot{x} = f(x, t, t - \tau)$

$$\dot{x} = ax \qquad x(t) = x_0 e^{-at}$$

$$\dot{x} = \frac{3}{4}\pi x_{t-2} \qquad x(t) = sin(\frac{3}{4}\pi t)$$

随机动力系统

Stochastic differential equation

$$dx = a(x,t)dt + b(x,t)dW$$

高斯白噪声

$$\frac{dW}{dt} = \xi(t), \ \langle \xi(t)\xi(t') \rangle = \delta(t - t')$$

Winer过程的微分过程

功率谱是一个常数

Fokker Planck Equation:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[a(x,t)p] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b^2(x,t)p]$$

Ornstein–Uhlenbeck process

$$\frac{dx}{dt} = -\frac{x}{\tau} + \sqrt{\frac{2\sigma^2}{\tau}}\xi(t)$$

Langevin equation(朗之万方程)

$$\frac{dx}{dt} = f(x, \mu) \qquad \text{参数}\mu = \langle\mu\rangle + \eta(t)$$

$$\frac{dx}{dt} = f(x, \langle\mu\rangle) + \eta(t)$$

$$\frac{dx}{dt} = f(x, \langle\mu\rangle) + g(x)\eta(t)$$

微分动力系统解的性质

解的存在和唯一性

The Existence and Uniqueness Theorem. *Consider the initial value problem*

$$X' = F(X), \quad X(0) = X_0$$

where $X_0 \in \mathbb{R}^n$. Suppose that $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then there exists a unique solution of this initial value problem. More precisely, there exists $a > 0$ and a unique solution

$$X: (-a, a) \to \mathbb{R}^n$$

of this differential equation satisfying the initial condition

$$X(0) = X_0.$$

Example1：

$$\dot{x} = \begin{cases} 1 & (x < 0) \\ -1 & (x \geq 0) \end{cases}$$

there is no solution that satisfies the initial condition $x(0) = 0$.

Example2: $\dot{x} = 3x^{2/3}$ with initial condition: x(0)=0

Multiple solutions:

$x(t)=0$

$x(t)=t^3$

$x(t,t')=0$ if $t'<0$ and $x(t,t')=(t-t')^3$ if $t'>0$

$3X^{2/3}$ is not differentiable at point 0

解对初值的连续依赖性

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. A function $F\colon \mathcal{O} \to \mathbb{R}^n$ is said to be *Lipschitz* on $\mathcal{O}$ if there exists a constant K such that

$$|F(Y) - F(X)| \leq K|Y - X|$$

for all $X, Y \in \mathcal{O}$. We call K a *Lipschitz constant* for F.

Theorem. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and suppose $F\colon \mathcal{O} \to \mathbb{R}^n$ has Lipschitz constant K. Let $Y(t)$ and $Z(t)$ be solutions of $X' = F(X)$ which remain in $\mathcal{O}$ and are defined on the interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$, we have*

$$|Y(t) - Z(t)| \leq |Y(t_0) - Z(t_0)| \exp(K(t - t_0)).$$

The solutions Y (t) and Z(t) start out close together, then they remain close together for t near t_0.

Corollary. (Continuous Dependence on Initial Conditions) *Let $\phi(t, X)$ be the flow of the system $X' = F(X)$ where F is C^1. Then ϕ is a continuous function of X.*

解对参数的连续依赖性

Theorem. (Continuous Dependence on Parameters) *Let* $X' = F_a(X)$ *be a system of differential equations for which* F_a *is continuously differentiable in both* X *and* a*. Then the flow of this system depends continuously on* a *as well as* X.

https://en.wikipedia.org/wiki/Dynamical_system

A dynamical system is a manifold M called the phase (or state) space endowed with a family of smooth evolution functions Φ^t that for any element of $t \in T$, the time, map a point of the phase space back into the phase space. The notion of smoothness changes with applications and the type of manifold. There are several choices for the set T. When T is taken to be the reals, the dynamical system is called a flow; and if T is restricted to the non-negative reals, then the dynamical system is a semi-flow. When T is taken to be the integers, it is a cascade or a map; and the restriction to the non-negative integers is a semi-cascade

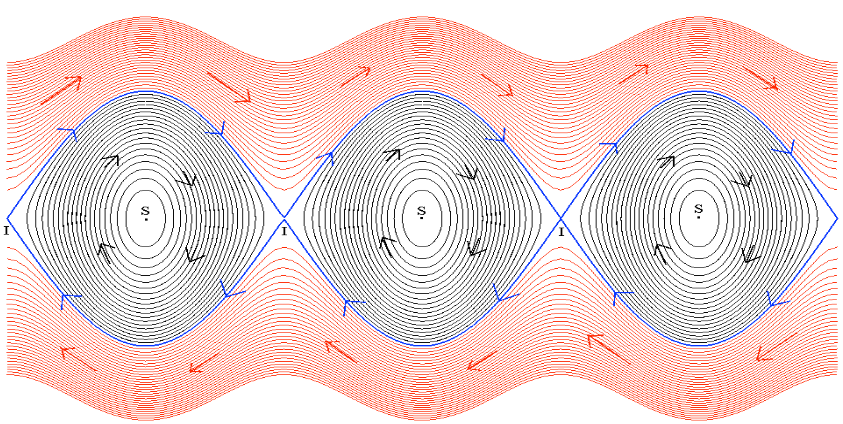
http://mathworld.wolfram.com/DynamicalSystem.html

A means of describing how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the reals or the integers on another object (usually a manifold). When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system.

https://en.wikipedia.org/wiki/Flow_(mathematics))

In mathematics, a **flow** formalizes the idea of the motion of particles in a fluid. Flows are ubiquitous in science, including engineering and physics. The notion of flow is basic to the study of ordinary differential equations. Informally, a flow may be viewed as a continuous motion of points over time. More formally, a flow is a group action of the real numbers on a set.

The idea of a vector flow, that is, the flow determined by a vector field, occurs in the areas of differential topology, Riemannian geometry and Lie groups. Specific examples of vector flows include the geodesic flow, the Hamiltonian flow, the Ricci flow, the mean curvature flow, and the Anosov flow. Flows may also be defined for systems of random variables and stochastic processes, and occur in the study of ergodic dynamical systems. The most celebrated of these is perhaps the Bernoulli flow.



Flow in phase space specified by the differential equation of a pendulum. On the x axis, the pendulum position, and on the y one its speed.

动力系统局部几何性质

两个基本问题

1. 搞清楚轨道在M上的几何结构，局部性质和全局性质。

2. 研究其时间过程，暂态过程以及渐近行为（时间极限行为）。

Why geometric view

$$\dot{x} = \sin x$$

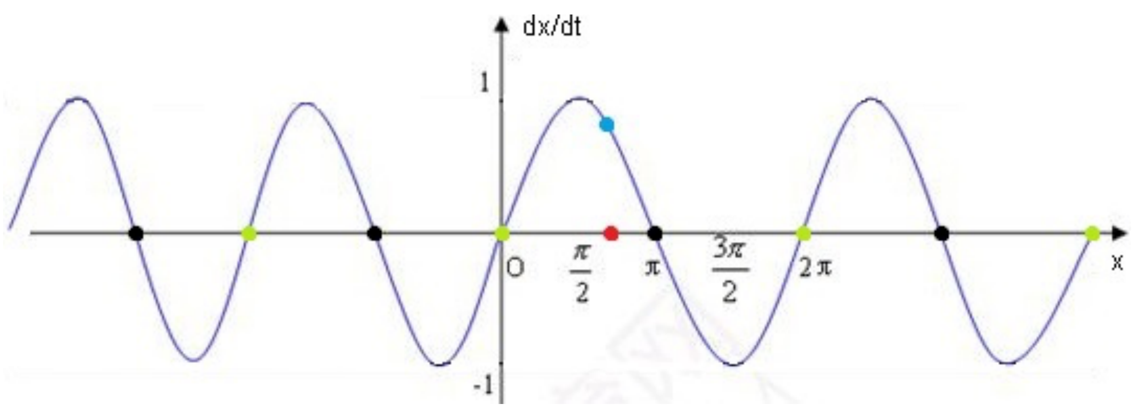
Analytical way:

$$dt = \frac{dx}{\sin x}$$

$$t = \int \csc x \, dx = -\ln\left|\csc x + \cot x\right| + C$$

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$$

Geometric view



定义（常点、奇点）：

动力系统$\varphi: \mathbb{R} \times M \to M$，若$x \in M$，使$\varphi^t(x) = x$ （$t \in \mathbb{R}$），则x是该动力系统的**不动点**。（奇点、**平衡点**、**定态点**、驻点、零点或临界点）

奇点之外的点都是常点，亦即通常遇到的点。

若$\varphi^t(\bar{x}) = \bar{x}$是一微分动力系统，则$\frac{d}{dt}\varphi^t(\bar{x})\,|_{t=0} = 0$ $\quad \dfrac{dx}{dt} = f(x) \quad f(\bar{x}) = 0$

若$f(\bar{x}) \neq 0$，那么$\bar{x}$称作常点

映射f构成的离散动力系统，若$f(\bar{x}) \neq \bar{x}$，则$\bar{x}$称为常点

导算子： 函数$f:\mathbb{R}^n\to\mathbb{R}^n$在点$a$可微是指存在线性算子$A:\mathbb{R}^n\to\mathbb{R}^n$，使得$f(a+h)-f(a)=Ah+o(\|h\|)\ (h\to 0)$成立。$A$称作$f(x)$在$a$处的导算子，记作$Df(a)$.

在一维系统中，导算子就是通常的导数；在高维系统中，导算子是雅可比算子，对高维系统的映射f 导算子 Df是雅可比矩阵：$Df=[\frac{\partial f_i}{\partial x_j}]$

对连续动力系统φ^t，它在$\bar{x}$点附近可近似表达为：

$$\varphi^t(\bar{x}+x)=\varphi^t(\bar{x})+D\varphi^t(\bar{x})x$$

其中$D\varphi^t(\bar{x})$是导算子，并且该动力系统对应于微分方程：$\frac{d}{dt}\varphi^t(x)=f(\varphi^t(x))$.

$$
\begin{aligned}
\frac{d}{dt}(D\varphi^t(\bar{x})x) &= \frac{d}{dt}[\lim_{s\to 0}\frac{1}{s}\{\varphi^t(\bar{x}+sx)-\varphi^t(\bar{x})\}] \\
&= \lim_{s\to 0}\frac{d}{dt}[\frac{1}{s}\{\varphi^t(\bar{x}+sx)-\varphi^t(\bar{x})\}] \\
&= \lim_{s\to 0}\frac{1}{s}[\frac{d}{dt}\varphi^t(\bar{x}+sx)-\frac{d}{dt}\varphi^t(\bar{x})] \\
&= \lim_{s\to 0}\frac{1}{s}(f[\varphi^t(\bar{x}+sx)]-f[\varphi^t(\bar{x})]) \\
&= \lim_{s\to 0}\frac{1}{s}(f(\varphi^t(\bar{x})+D\varphi^t(\bar{x})sx)-f[\varphi^t(\bar{x})]) \\
&= \lim_{s\to 0}\frac{1}{s}(f(\varphi^t(\bar{x}))+Df(\varphi^t(\bar{x})\cdot D\varphi^t(\bar{x})sx)-f[\varphi^t(\bar{x})]) \\
&= Df(\varphi^t(\bar{x}))\cdot D\varphi^t(\bar{x})x
\end{aligned}
$$

$$\frac{d}{dt}[D\varphi^t(\bar{x})x] = Df(\bar{x})D\varphi^t(\bar{x})x$$

$$\Rightarrow \quad D\varphi^t(\bar{x})x = e^{Df(\bar{x})t}x$$

对于常点：

$$\varphi^t(\bar{x}+x) = \varphi^t(\bar{x}) + e^{Df(\bar{x})t}x$$

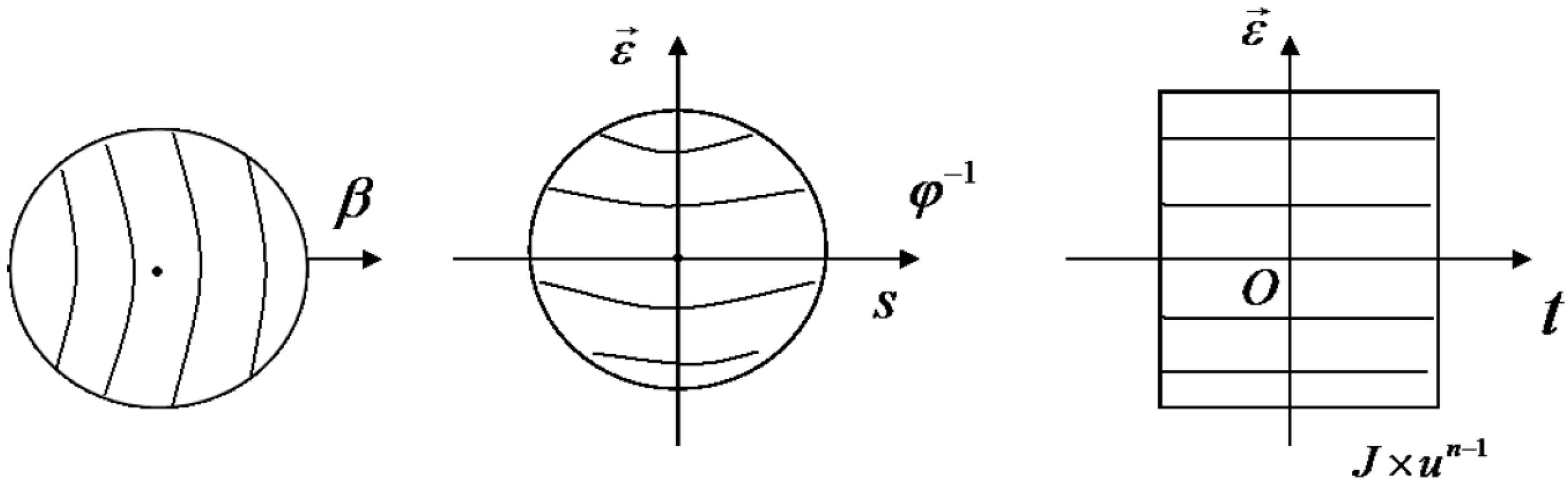
对于奇点：

$$\varphi^t(\bar{x}+x) = \bar{x} + e^{Df(\bar{x})t}x$$

常点直化定理:

设有定义在开集$G\in\mathbb{R}^n$上的C^r微分动力系统φ^t，$\bar{x}\in G$是φ^t 的一个常点，则在$\bar{x}$的邻域$U(\bar{x})$内，及$U(\bar{x})$上存在C^r 的微分同胚α，该同胚映射将$U(\bar{x})$内的流对应为$\mathbb{R}^n$内原点邻域的一簇平行直线段。

简单的说明: 微分动力系统φ^t可以表达为$\frac{d}{dt}x=f(x)$，$\tilde{x}$ 是φ^t 的常点，说明$\frac{d}{dt}x\,|_{x=\tilde{x}}=f(\tilde{x})\neq 0$，($\vec{0}$ 是一个列向量)，则可以通过平移变换$\tilde{x}\to$原点，通过非奇异的线性变换将$f(\tilde{x})$转化为$(1,0,\ldots,0)^T=(1,\vec{0})^T,\vec{0}$ 是$n-1$ 维向量。



对于奇点，根据奇点导算子的特征根:$\lambda_1, \ldots, \lambda_n$，可以将奇点分为以下几类:

非双曲平衡点: $\exists i, \quad Re\,\lambda_i = 0$

双曲平衡点(Hyperbolic): $\forall i, \quad Re\,\lambda_i \neq 0$

焦点 $Im\lambda_i \neq 0$

结点

鞍点 $\lambda_i > 0,\ \lambda_j < 0$

汇点: Sink $\forall i, \ Re\lambda_i < 0$

源点: Source $\forall i, \ Re\lambda_i > 0$

Hartman定理： $\mathbb{R}^n$上的动力系统$\varphi^t(x)$满足$\dot{x} = f(x)$，$\bar{x}$是$\varphi^t(x)$的双曲平衡点，则存在$\bar{x}$的邻域$U \subset \mathbb{R}^n$和同胚映射$h: U \to h(U) \subset \mathbb{R}^n$，使得：$h(\bar{x}) = 0$，且$h(\varphi^t(x)) = e^{Df(\bar{x})t} \cdot h(x)$

平衡点的稳定流形定理：

$\bar{x}$是M上连续动力系统$\varphi^t(x)$的双曲平衡点，且$\dot{x} = f(x)$，则存在稳定和不稳定流形 $W^s_{loc}(\bar{x})$，$W^u_{loc}(\bar{x})$，且分别与 $\dot{x} = Df(\bar{x})x$的稳定与不稳定子空间 E^s和 E^u在$\bar{x}$相切。

特征向量：$V_1, \ldots, V_{n_s}$，$U_1, \ldots, U_{n_u}$，$W_1, \ldots, W_{n_c}$，对应特征根$Re\ \lambda_i < 0$，$Re\ \lambda_i > 0$，$Re\ \lambda_i = 0$

切空间： $E_s = Span\{V_1, \ldots, V_{n_s}\}$，$E_u = Span\{U_1, \ldots, U_{n_u}\}$，$E_c = Span\{W_1, \ldots, W_{n_c}\}$.

稳定流形： $W^s_{loc}(\bar{x}) = \{x \in U;\ \ \bar{x} \in U \subset M,\ \ \varphi^t(x) \to \bar{x},\ t \to \infty$ 且 $\varphi^t(x) \in U,\ \forall t\}$

非稳定流形： $W^u_{loc}(\bar{x}) = \{x \in U;\ \ \bar{x} \in U \subset M,\ \ \varphi^t(x) \to \bar{x},\ t \to -\infty$ 且 $\varphi^t(x) \in U,\ \forall t\}$

Example： a planar system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$, $\lambda_i \neq 0$;

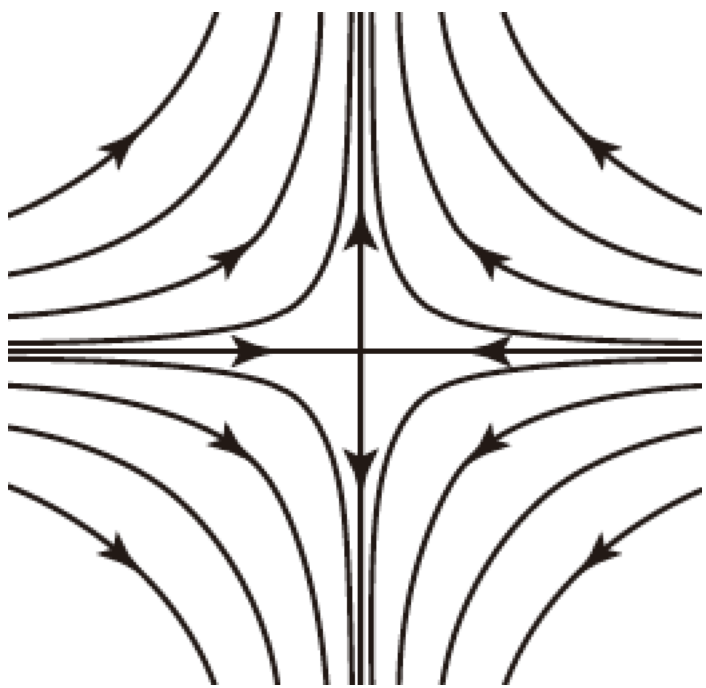
1. $\lambda_1 < 0 < \lambda_2$;   2. $\lambda_1 < \lambda_2 < 0$;   3. $0 < \lambda_1 < \lambda_2$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

An eigenvector corresponding to λ_1 is $(1, 0)$ and to λ_2 is $(0, 1)$

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Phase portrait:



$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

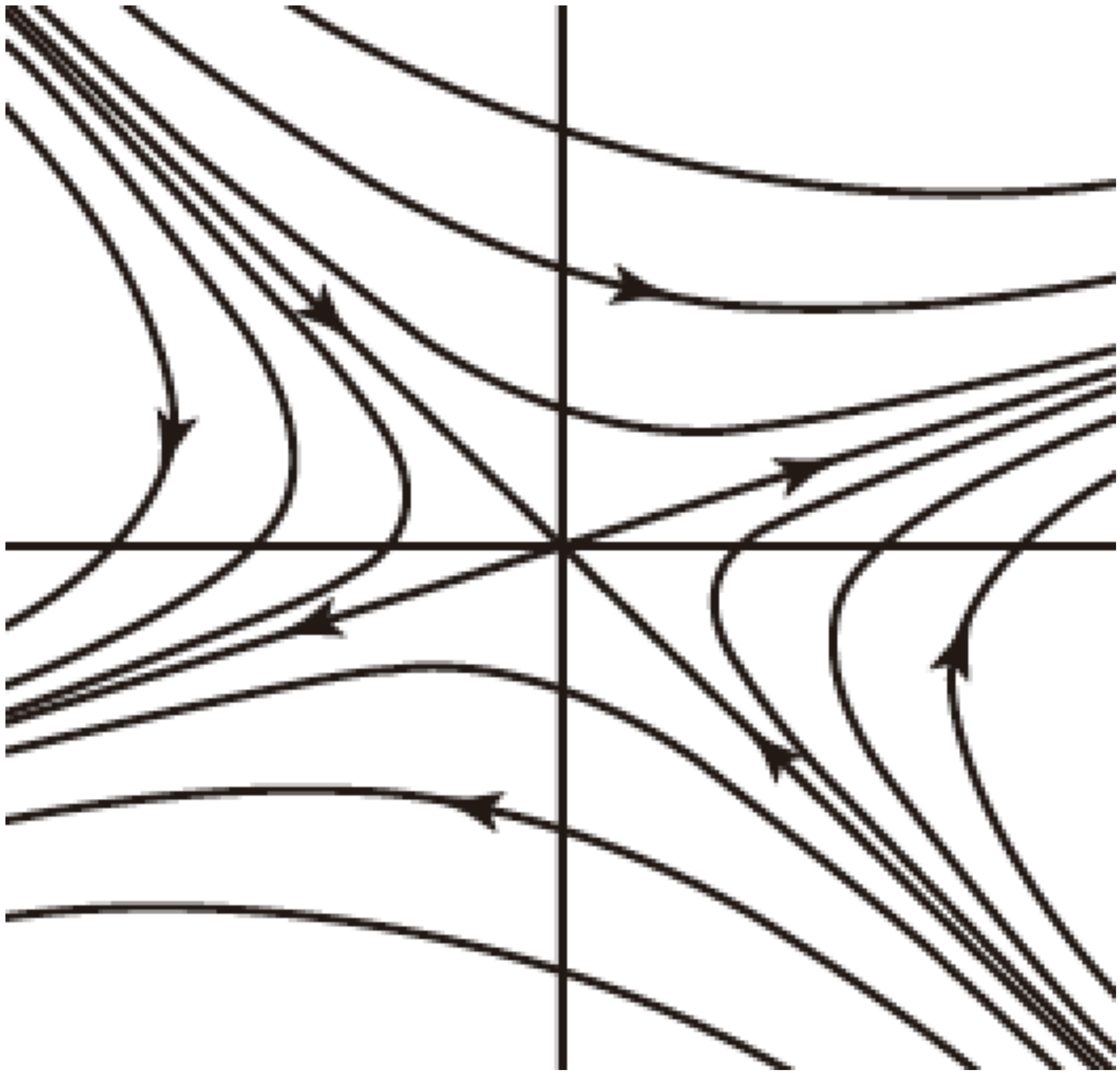
the eigenvalues of A are ± 2.

An eigenvector corresponding to λ_1 is $(3, 1)$,and$(1, -1)$ to $\lambda = -2$

Unstable line: $X_1(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Stable line: $X_2(t) = \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$X(t) = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \lambda_1 < \lambda_2 < 0$$

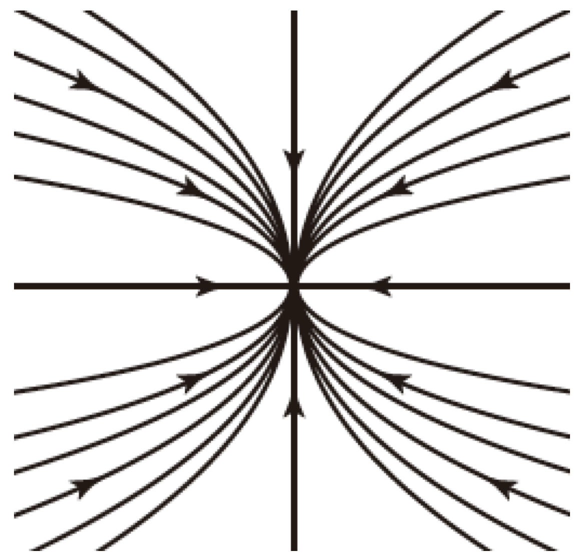
$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t}$$

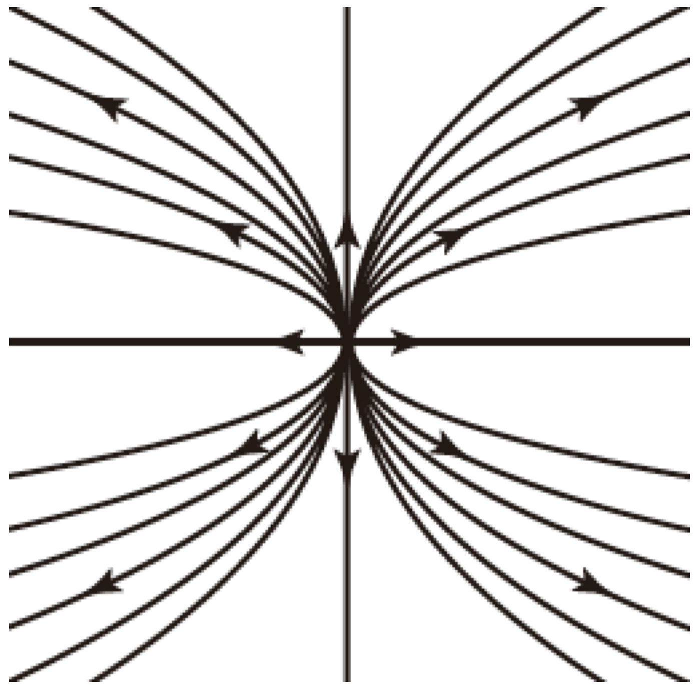
Since $\lambda_2 - \lambda_1 > 0$, it follows that these slopes approach $\pm\infty$

call λ_1 the stronger eigenvalue

λ_2 the weaker eigenvalue



$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{satisfies } 0 < \lambda_2 < \lambda_1$$



$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 + \beta^2 = 0$;

the eigenvalues are the imaginary numbers $\pm i\beta$.

the eigenvector corresponding to $\lambda = i\beta$ satisfies

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

find a complex eigenvector $(1, i)$ and a complex solution

$$X(t) = e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Using Euler's formula: $e^{i\beta t} = \cos\beta t + i\sin\beta t$

$$X(t) = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ i(\cos\beta t + i\sin\beta t) \end{pmatrix} = \begin{pmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{pmatrix}$$

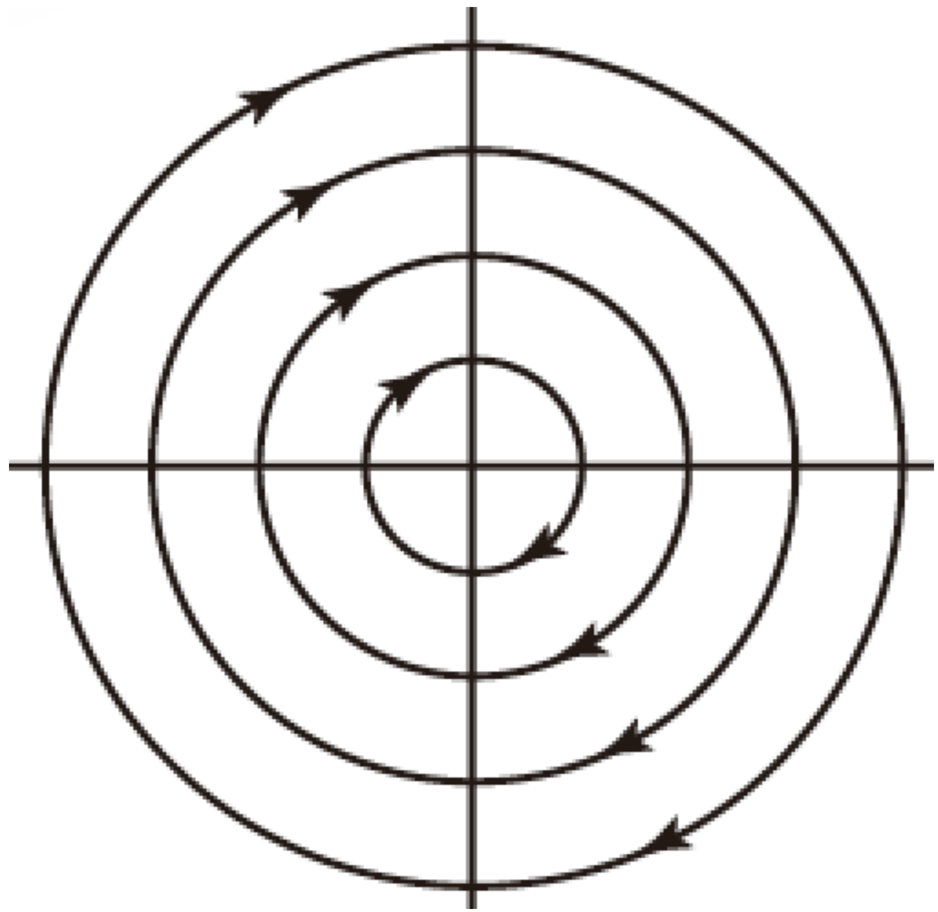
$$X(t) = X_{\text{Re}}(t) + iX_{\text{Im}}(t)$$

$$X_{\text{Re}}(t) = \begin{pmatrix} \cos\beta t \\ -\sin\beta t \end{pmatrix}, \; X_{\text{Im}}(t) = \begin{pmatrix} \sin\beta t \\ \cos\beta t \end{pmatrix}$$

Where $X_{Re}(t)$ and $X_{Im}(t)$ are solutions of the original system

$$\begin{aligned} \underline{X'_{\text{Re}}(t)} + \underline{iX'_{\text{Im}}(t)} = X'(t) &= AX(t) \\ &= A(X_{\text{Re}}(t) + iX_{\text{Im}}(t)) \\ &= \underline{AX_{\text{Re}}} + \underline{iAX_{\text{Im}}(t)}. \end{aligned}$$

$$X(t) = c_1 X_{\text{Re}}(t) + c_2 X_{\text{Im}}(t)$$

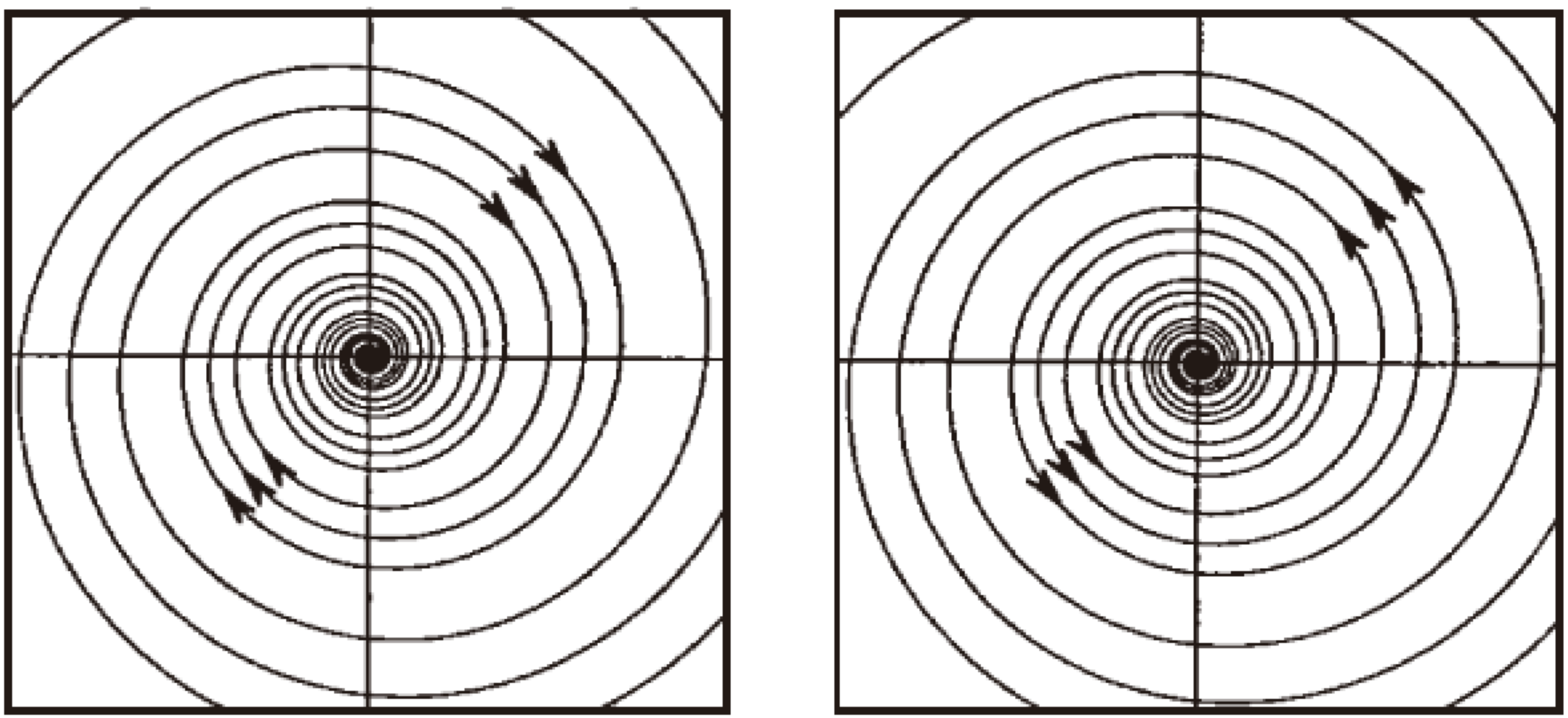


$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

the eigenvalues are $\lambda = \alpha \pm i\beta$ $(1, i)$ is an eigenvector.

$$\begin{aligned} X(t) &= e^{(\alpha+i\beta)t} \begin{pmatrix} 1 \\ i \end{pmatrix} \\ &= e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + i e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix} \\ &= X_{\mathrm{Re}}(t) + i X_{\mathrm{Im}}(t). \end{aligned}$$

$$X(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}$$

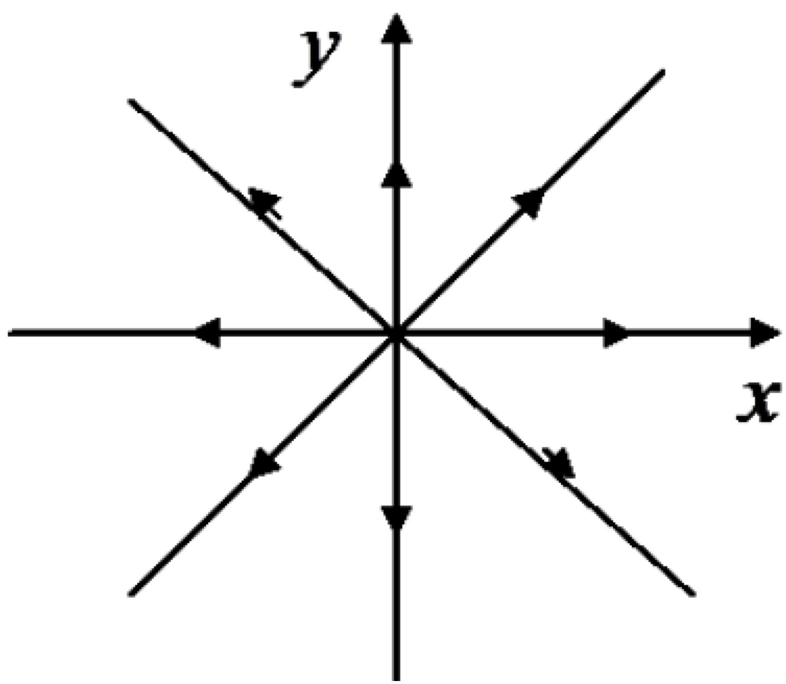


Phase portraits for a spiral sink and a spiral source.

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Every nonzero vector is an eigenvector since $AV = \lambda V$ for any $V \in \mathbb{R}^2$

$$X(t) = \alpha e^{\lambda t} V$$



$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

There is only one linearly independent eigenvector given by (1, 0).

$$X_1(t) = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To find other solutions, note that the system can be written：

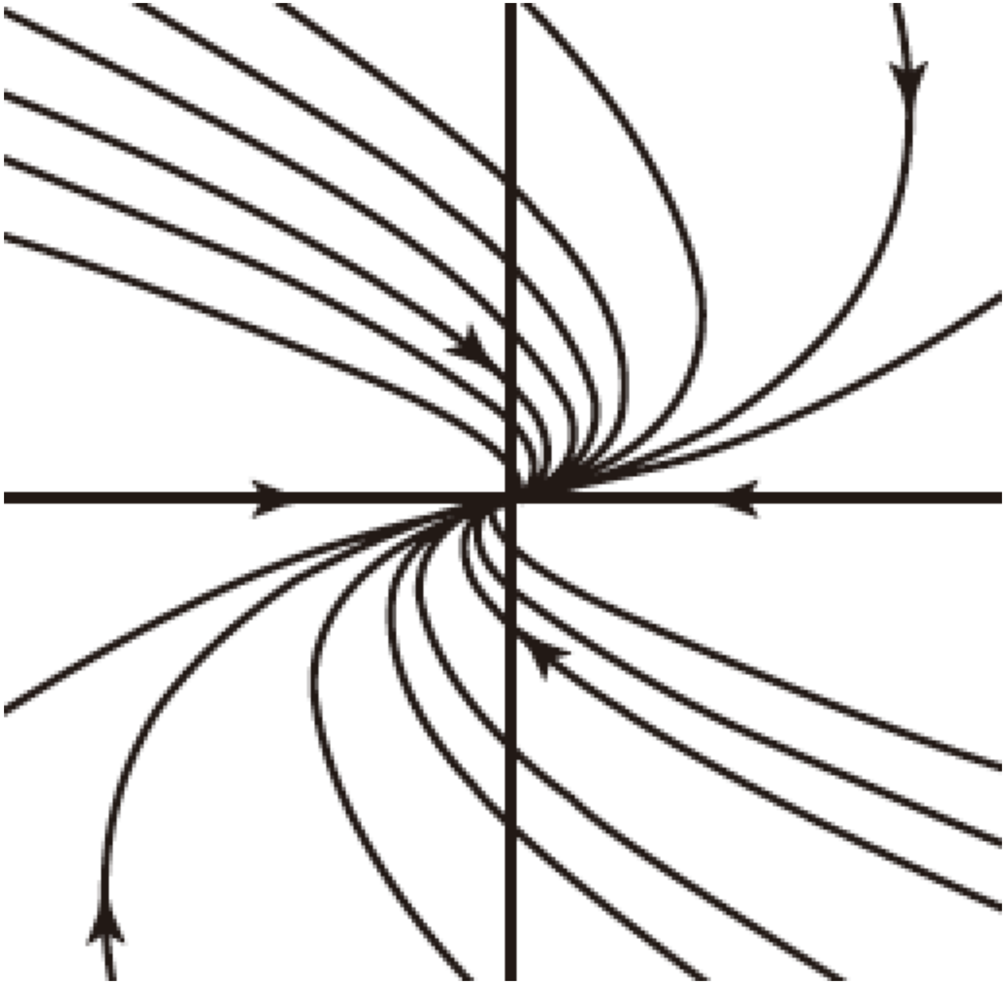
$$\begin{aligned} \dot{x} &= \lambda x + y \\ \dot{y} &= \lambda y \end{aligned}$$

we must have

$$y(t) = \beta e^{\lambda t}$$

$$\dot{x} = \lambda x + \beta e^{\lambda t}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$



利用以下公式重新计算上面例子中的所有结果（更加简洁）

$$e^{At} = (I + At + \frac{1}{2!}A^2t^2 + \frac{t^3}{3!}A^3 + \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n$$

Example：

$$A = \begin{pmatrix} \lambda & 0 \\ \xi & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{At}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}\left(\xi + \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} t\right)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = e^{\lambda t}\begin{pmatrix} x_0 \\ \xi t x_0 + y_0 \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} x_0 \cos \omega t - y_0 \sin \omega t \\ x_0 \sin \omega t + y_0 \cos \omega t \end{pmatrix} = e^{\lambda t} \begin{pmatrix} q cos(\omega t + \varphi) \\ q sin(\omega t + \varphi) \end{pmatrix}$$

$$\begin{aligned} e^{\begin{pmatrix} 0 & -\omega t \\ \omega t & 0 \end{pmatrix}} &= \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & -\omega t \\ \omega t & 0 \end{pmatrix}^n \\ &= I + \begin{pmatrix} 0 & -\omega t \\ \omega t & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -(\omega t)^2 & 0 \\ 0 & -(\omega t)^2 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} 0 & -(\omega t)^3 \\ (\omega t)^3 & 0 \end{pmatrix} \\ &\quad + \frac{1}{4!} \begin{pmatrix} (\omega t)^4 & 0 \\ 0 & (\omega t)^4 \end{pmatrix} + \frac{1}{5!} \begin{pmatrix} 0 & (\omega t)^5 \\ -(\omega t)^5 & 0 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \end{aligned}$$

对高维系统，情况与二维平面系统类似。只要是双曲平衡点，其邻域内的轨道都可以于一个线性动力系统。

渐近行为（asymptotic behavior ）

关注的第二个问题：动力系统在时间无穷后的行为，也就是渐进行为。

1. 极限点、极限集

φ^t是M上的动力系统，$\bar{x} \in M$称作轨道$\varphi^t(x_0)$的ω极限点，如果∃序列$t_n \to +\infty$，使$\varphi^{t_n}(x_0) \to \bar{x}_0$。$\varphi^t(x_0)$的所有$\omega$极限点组成的集合为$\varphi^t(x_0)$的极限集，记作$L_\omega(x_0)$. $L_\omega = \bigcup_{x_0 \in M} L_\omega(x_0)$称作$\varphi^t$的$\omega$极限集。当$t \to -\infty$时，相应地有$\alpha$极限集$L_\alpha(x_0)$

$$L(x_0) = L_\alpha(x_0) \bigcup L_\omega(x_0) \qquad L = \bigcup_{x_0 \in M} L(x_0)$$

E.g.1

If $\hat{\boldsymbol{x}}$ is an equilibrium $\omega(\hat{\boldsymbol{x}}) = \alpha(\hat{\boldsymbol{x}}) = \hat{\boldsymbol{x}}$.

E.g.2

$\dot{x} = -x, \quad x \in \mathbf{R}$

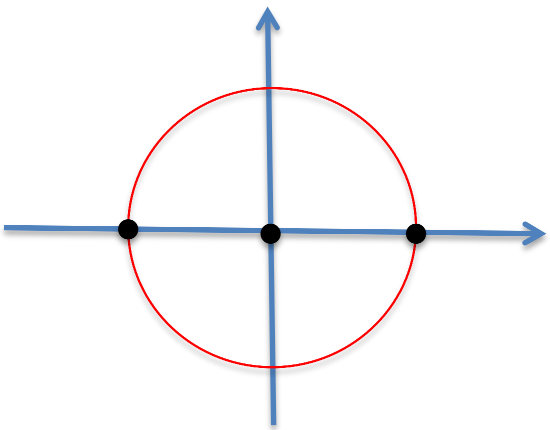
For the equilibrium $\hat{x} = 0$ we have that $\alpha(0) = \omega(0) = 0$. For any other point $x \in \mathbf{R} \setminus \{0\}$ we obviously have

$$\omega(x) = 0, \quad \alpha(x) = \emptyset.$$

E.g.3

$$\dot{r} = r(a - r),$$
$$\dot{\theta} = \sin^2\theta + (r - a)^2$$

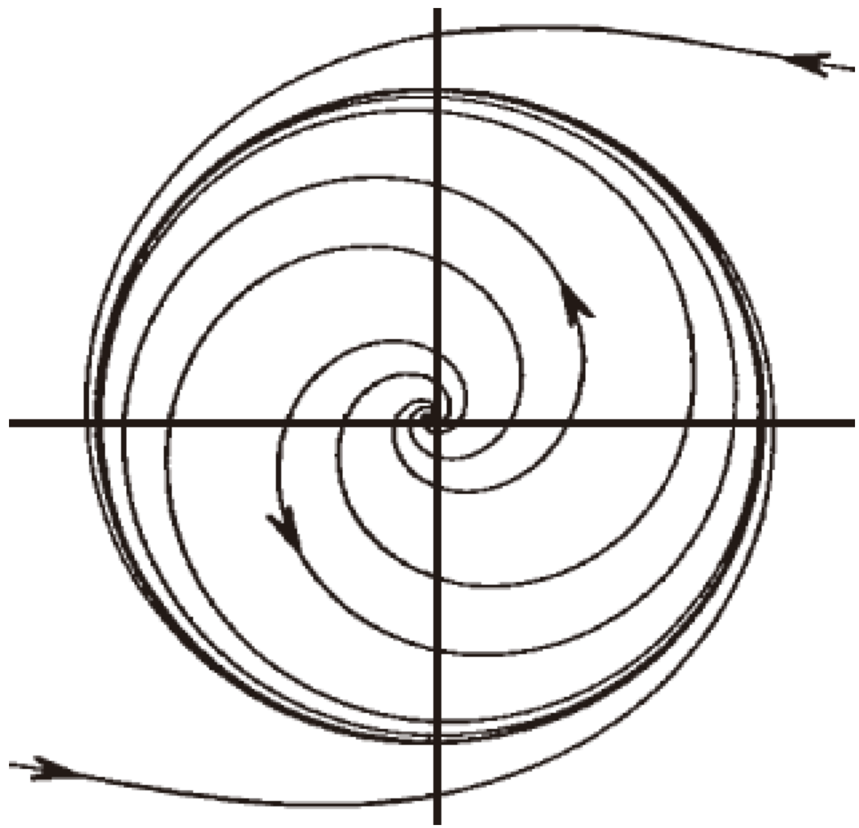
three equilibria $\hat{r}_0 = 0$, $(\hat{r}, \hat{\theta}) = (a, 0)$, and $(\hat{r}, \hat{\theta}) = (a, \pi)$



The **ω**-limi set for most of the initial conditions represent the closed curve composed by two equilibria on the circle, and the heteroclinic trajectories connecting them

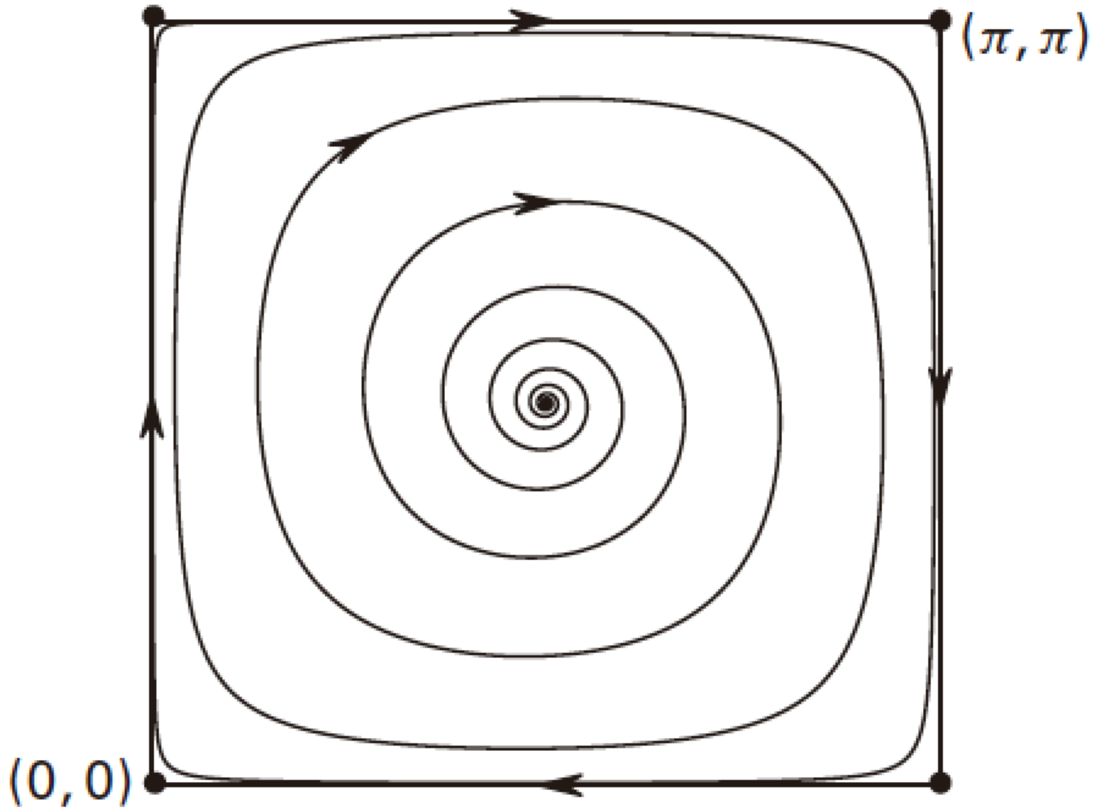
E.g.4

$$\begin{aligned} \dot{r} &= r - r^3 \\ \dot{\theta} &= 1 \end{aligned}$$



E.g.5

$$\begin{aligned}\dot{x} &= \sin x(-0.1\cos x - \cos y)\\ \dot{y} &= \sin y(\cos x - 0.1\cos y)\end{aligned}$$

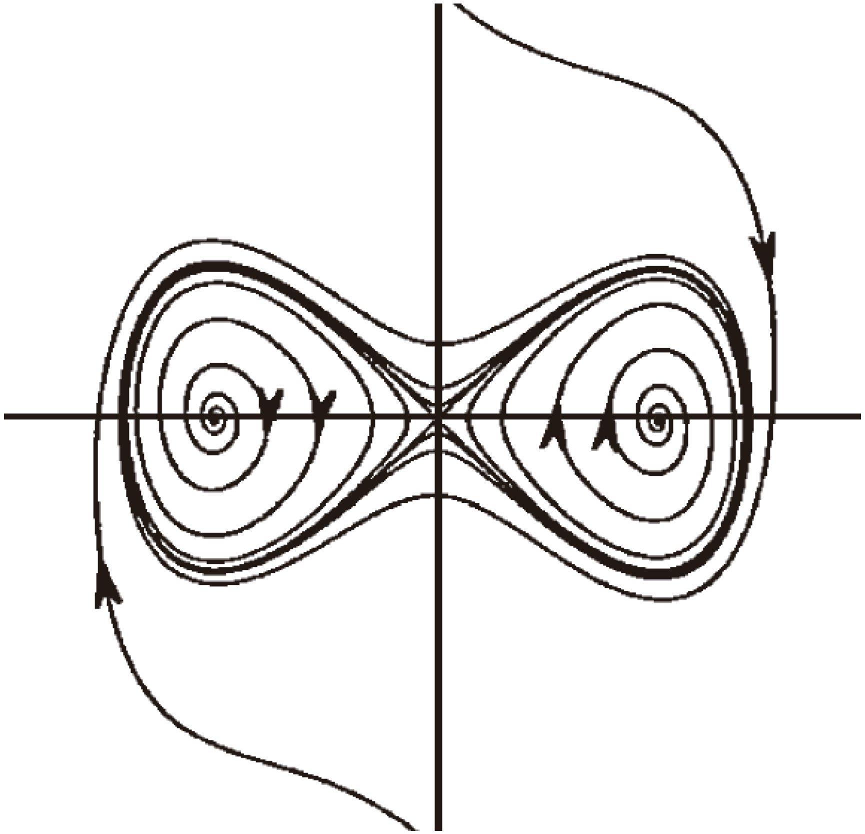


The **ω**-limit set of any solution emanating from the source at **(π/2, π/2)** is the square bounded by the four equilibria and the heteroclinic solutions.

E.g.6

$$x' = -y - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)(x^3 - x)$$

$$y' = x^3 - x - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)y.$$



Three equilibria: at (0, 0), (−1, 0), and (1, 0). The origin is saddle, and there are two homoclinic solution

2. **非游荡集（nonwandering set）**

对$x \in M$的任意邻域$U \subset M$都存在$T > 0$，使得$\varphi^T(U) \bigcap U \neq \emptyset$. 则称$x$是动力系统$\varphi^t$的非游荡点。所有非游荡点组成的集合称作非游荡集，记作Ω.

3. **周期点**

$x \in M,\ \exists T > 0,\ \ \varphi^{t+T}(x) = \varphi^t(x)$ 对 $\forall t \in \mathbb{R}$

4. **回归点**

$x \in U \subset M,\ \ \forall t > 0,\ \ \varphi^t(x) \in U$ 回到邻域无穷次。

若记不动点集为F，周期点集为P，回归点集为R，则有

$$F \subset P \subset R \subset L \subset \Omega$$

5. 不变集（invariant set）

集合A称作不变集，若 $\forall x \in A$，对 $\forall t \in \mathbb{R}$ 有 $\varphi^t(x) \in A$. 有$t > 0$，则称正不变集，$t < 0$，则称负不变集。

6. 吸引集(abstracting set)

集合A称作吸引集，如果A是闭不变集，且存在开集$U \supset A$，使得 $\forall x \in U$和 $t > 0$，当$t \to +\infty$ 时，$\varphi^t(U) = A$. 若U是整个集合M，则称集合A为整体吸引集。相应地，集合 $\bigcup_{t \le 0} \varphi^t(U)$称作$A$的吸引域。

若吸引集A是拓扑可迁的，则A称作**吸引子（abstractor）**.

一个闭不变集A是拓扑可迁，若对任意开集 $U, V \subset A$,$\exists t > 0$，使$\varphi^t(U) \bigcap V \ne \emptyset$.

一些性质：

1.a) 任一整轨线是一个不变集。

b) 任一不变集由一系列整轨线组成。

2. 极限集是闭集。（有限维空间）

3. 极限集是不变集，因此是由整轨线组成。

4. ω_α极限集为空($|\varphi^t(x_0)| \to +\infty$)的充要条件为 $t \to \pm\infty$ 时，轨线趋向无穷。

5. $L_\omega(x)$为单点集的充要条件是

$$\lim_{t\to\infty} \varphi^t(x_0) = \bar{x} \qquad (\lim_{t\to-\infty} \varphi^t(x_0) = \bar{x})$$

6. A是一个闭的不变集（特别地A为极限集），若$x_0 \in A$，则$L_\omega(x_0) \subset A$且$L_\alpha(x_0) \subset A$.

Theorem. (Poincaré-Bendixson) *Suppose that Ω is a nonempty, closed and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

动力系统稳定性初步

（轨道道稳定性和状态稳定性）

对于现实的动力系统，并不是动力系统所有的解都能能够被轻易观测到。有些解存在但几乎不会被观测到，只有具有某种稳定性的结构和结果才能被观测到。

轨道稳定性（Poncáre稳定性）

定义于M上的动力系统φ^t称作**轨道稳定**的，如果：

$\forall \varepsilon > 0$，$\exists \delta(\varepsilon) > 0$，$M$上的距离 $\|x^{01} - x^{02}\| < \delta$，则$\|Orb_{\varphi}(x^{01}) - Orb_{\varphi}(x^{02})\| < \varepsilon$

或者：

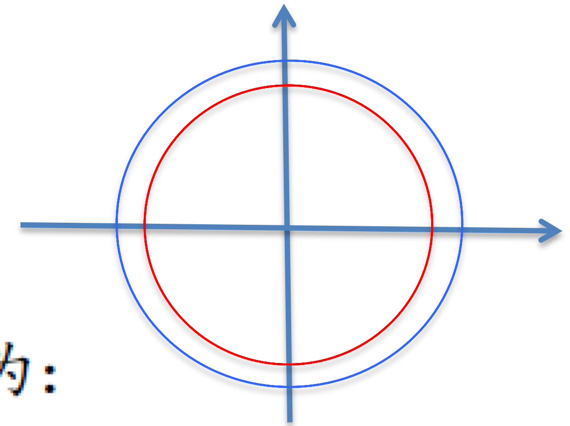
$\forall \varepsilon > 0$，$\exists \delta(\varepsilon) > 0$，$\exists \tau \geq t_0$时，当$\|\varphi^t(x^{01}) - \varphi^{\tau}(x^{02})\| > \delta$时(其中$x^{01}, x^{02}$为不同的初始值)，

存在映射T：$[t_1, t_2] \to [t_1, t_2]$使得$\|\varphi^t(x^{01}) - \varphi^{T(t)}(x^{02})\| < \varepsilon$

若$t \to \infty$时，$\|\varphi^t(x^{01}) - \varphi^{T(t)}(x^{02})\| \to 0$，则称**轨道渐近稳定**

$$\dot{\theta} = r$$

$$\dot{r} = 0$$



满足初始条件$\theta(t_0) = \theta_0 = 0, r(t_0) = c$的轨道为：

$$Orb_{\varphi}(x, y) = (x, y),\ \ x^2 + y^2 = c^2$$

$\forall \varepsilon > 0$ 取 $\delta = \varepsilon,\ \tau = t_0$，有：

$$\|\varphi^{t_0}(c, 0) - \varphi^{t_0}(c + \frac{1}{2}\delta, 0)\| = \|c - (c + \frac{1}{2}\delta)\| = \frac{1}{2}\delta < \delta$$

令 $T(t) = \frac{c}{c+\frac{1}{2}\delta}t$

$$\|\varphi^{t}(c, 0) - \varphi^{T(t)}(c + \frac{1}{2}\delta, 0)\| = \frac{1}{2}\delta < \varepsilon.$$

根据定义：该系统是轨道稳定的。

轨道稳定的含义是：如果两条轨道曾经有相近的点，那么就可以通过时间轴的变换使所有的点之间的距离不超过一定距离甚至完全重合。初略地说，就是两条轨道经过了相似的历史事件。

状态稳定性（Lyapunov稳定性）

定义于M上的φ^t，$x_0 \in M$，对$\forall \varepsilon > 0, \exists \delta > 0, T > 0, \forall t > T, \forall x \in U_{x_0}(\delta)$，即$x$在$x_0$的邻域或$\|x - x_0\| < \delta$，则 $\|\varphi^t(x_0) - \varphi^t(x)\| < \varepsilon$ 则称$\varphi^t(x_0)$具有Lyapunov稳定性，或$\varphi^t(x_0)$是Lyapunov稳定的。

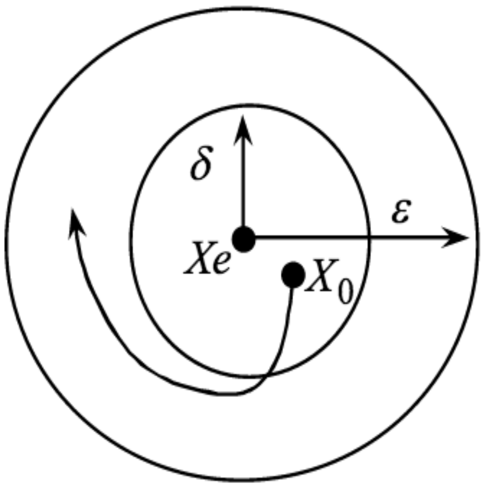
若$\|\varphi^t(x_0) - \varphi^t(x)\| \to 0$ 则称$\varphi^t(x_0)$具有Lyapunov渐近稳定性或Lyapunov渐近稳定的.

若$\bar{x}$是平衡点， $\forall \varepsilon > 0, \exists T > 0, \delta > 0$，当$\|\bar{x} - x\| < \delta, t > T$时:

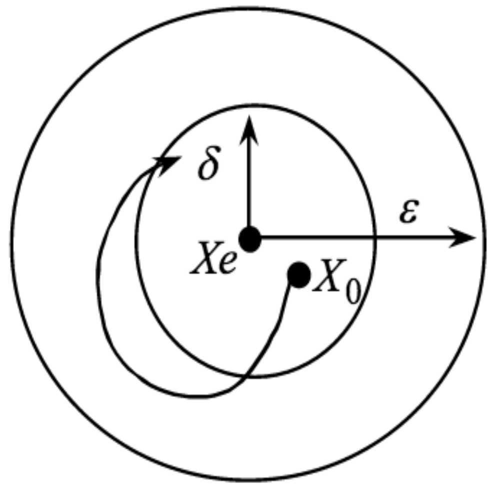
若$\|\varphi^t(\bar{x}) - \varphi^t(x)\| = \|\varphi^t(x) - \bar{x}\| < \varepsilon$，则$\bar{x}$称为稳定的。

若$\|\varphi^t(x) - \bar{x}\| \to 0$，则称为渐近稳定的。

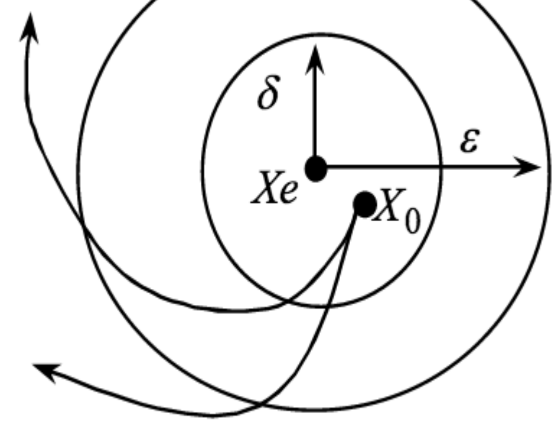
以上定义意味着：对平衡态Xe，首先选择一个域Σ(ε)，对应于每一个Σ(ε),必存在一个域Σ(δ), 使得当t趋于无穷时，始于Σ(δ)的轨迹总不脱离域Σ(ε)。



稳定



渐近稳定



不稳定

ε是一个给定的稳定性指标，最终偏差小于它，则稳定；而δ则描述了稳定域的大小，即系统稳定的范围，或能忍受的最大干扰。当初始状态与平衡状态的偏差小于它时，则稳定。对于渐近稳定，实际上就是将稳定性指标取为ε=0，即要求系统最终状态完全回到某一状态，即以某一状态为极限点。

李雅普诺夫函数（V函数）

$x \in U(x^*)$，$V(x) \geq 0 (\leq 0)$且 $V(x^*) = 0$，$V \in C^1$，则V为x^*处的李雅普诺夫函数。

其中$U(x^*)$代表：x^*一点的邻域。

定理:

连续流$\varphi^t(x)$ (i.e. $\dot{x} = f(x)$)有定态解$\bar{x}$，且$\bar{x}$处有李氏函数V。

① 若 $x \in U(\bar{x})$, $V(x) \geq 0$，$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} \leq 0$，则$\bar{x}$是稳定的.

② 若$V(x) \geq 0$, $V(x) = 0 \Leftrightarrow x = \bar{x}$, $x \in U(\bar{x})$且 $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} < 0$，则$\bar{x}$渐近稳定。

E.g.1

$$\dot{x} = -x^3$$

$$V(x) = x^2$$

$$\frac{dV}{dt} = V'(x)\frac{dx}{dt} = 2x\frac{dx}{dt},$$

$$\frac{dV}{dt} = -2x^4.$$

E.g.2 具有非线性恢复力的振子

无阻尼： $m\ddot{x} + k(x + x^3) = 0$

$$\dot{x} = y$$
$$\dot{y} = -\frac{k}{m}(x + x^3)$$

系统有定态 $(\bar{x}, \bar{y}) = (0, 0)$

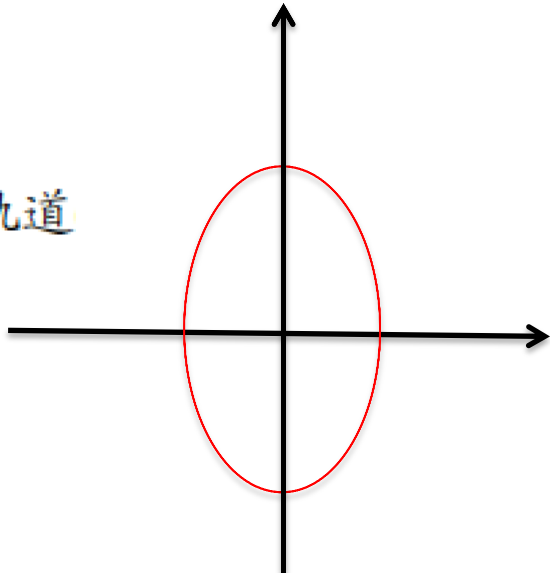
Lyapunov函数可取为系统总能量

$$V(x, y) = E(x, y) = 1/2my^2 + k(1/2x^2 + 1/4x^4) \geqslant 0$$

$$\begin{aligned}\frac{dE}{dt} &= \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} \\ &= kxy + kx^3y + my\dot{y} \\ &= ky(x + x^3) - ky(x + x^3) = 0 \leqslant 0\end{aligned}$$

于是(0, 0)是Lyapunov稳定的

$y^2 + \frac{k}{m}(x^2 + \frac{1}{2}x^4) = c^2$是该非线性回复力振子的轨道



有阻尼： $m\ddot{x}+\alpha\dot{x}+k(x+x^3)=0$

$$\begin{aligned}\dot{x}&=y\\ \dot{y}&=-\tfrac{k}{m}(x+x^3)-\tfrac{\alpha}{m}y\end{aligned}$$

若选用 $E(x,y)=\frac{1}{2}my^2+k(\frac{1}{2}x^2+\frac{1}{4}x^4)$

$$\frac{dE}{dt}=my(-\frac{k}{m}(x+x^3)-\frac{\alpha}{m}y)+k(x+x^3)y=-\alpha y^2\leqslant 0$$

$(0,0)$是稳定的，但无法判定$(0,0)$是否渐近稳定

$$V(x,y) \quad = \quad \frac{1}{2}my^2+k(\frac{1}{2}x^2+\frac{1}{4}x^4)+\beta(xy+\frac{1}{2}\frac{\alpha}{m}x^2)$$

$$\begin{aligned}\frac{dV}{dt} &= my\dot{y}+k(x+x^3)\dot{x}+\beta(x\dot{y}+\dot{x}y+\frac{\alpha}{m}x\dot{x})\\ &= (my+\beta x)\dot{y}+(k(x+x^3)+\beta y+\frac{\alpha}{m}x)\dot{x}\\ &= -(my+\beta x)(\frac{k}{}(x+x^3)+\frac{\alpha}{n}y)+(k(x+x^3)+\beta y+\frac{\alpha}{m}x)y\\ &= -(\alpha-\beta)y^2-\frac{\beta k}{m}(x^2+x^4)\end{aligned}$$

当$0<\beta<\alpha$时，$V(x,y)>0$且 $\frac{dV}{dt}<0$，$(0,0)$渐近稳定。

E.g.3

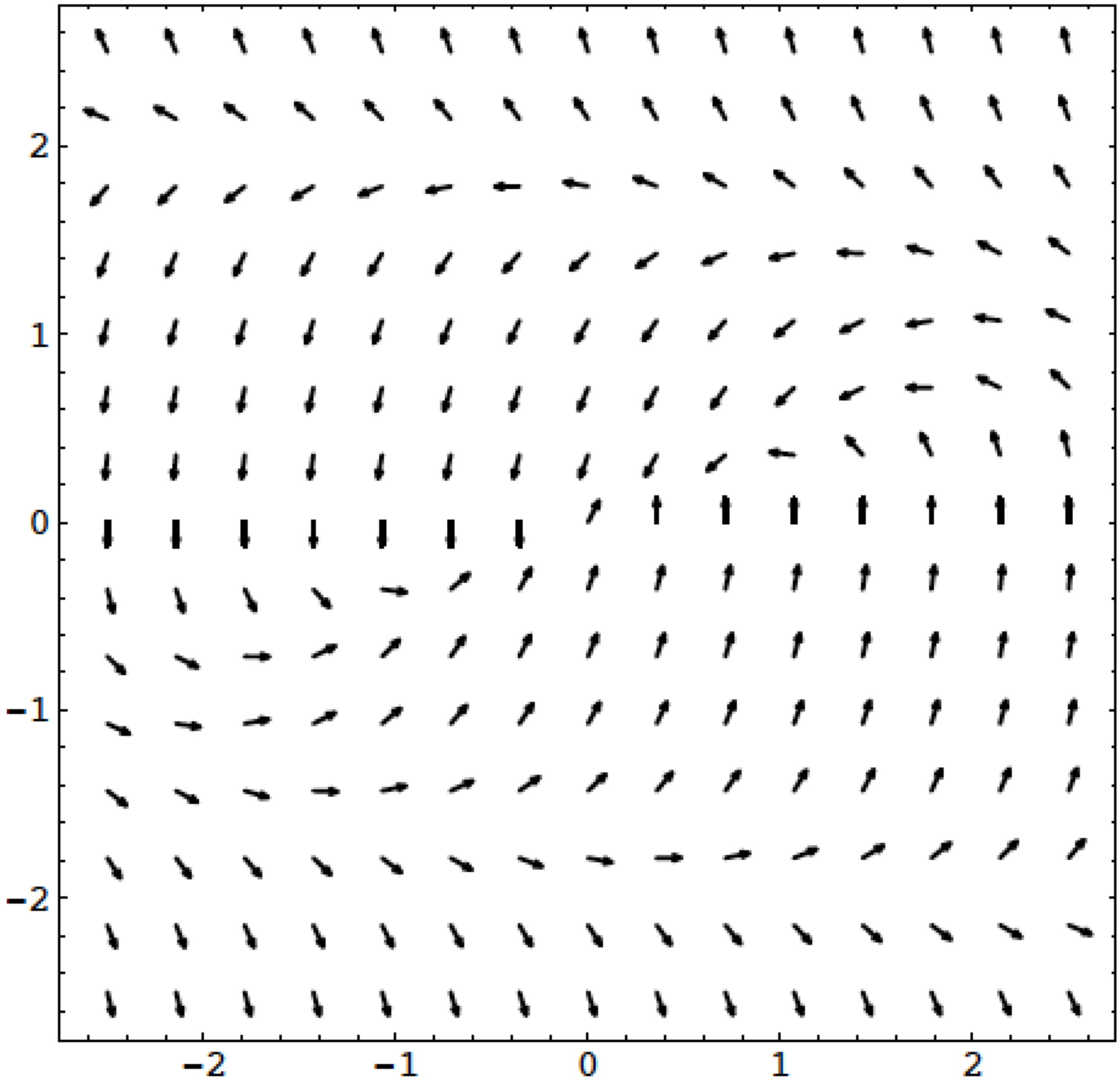
$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = x_1 + x_2^3 - 3x_2$$

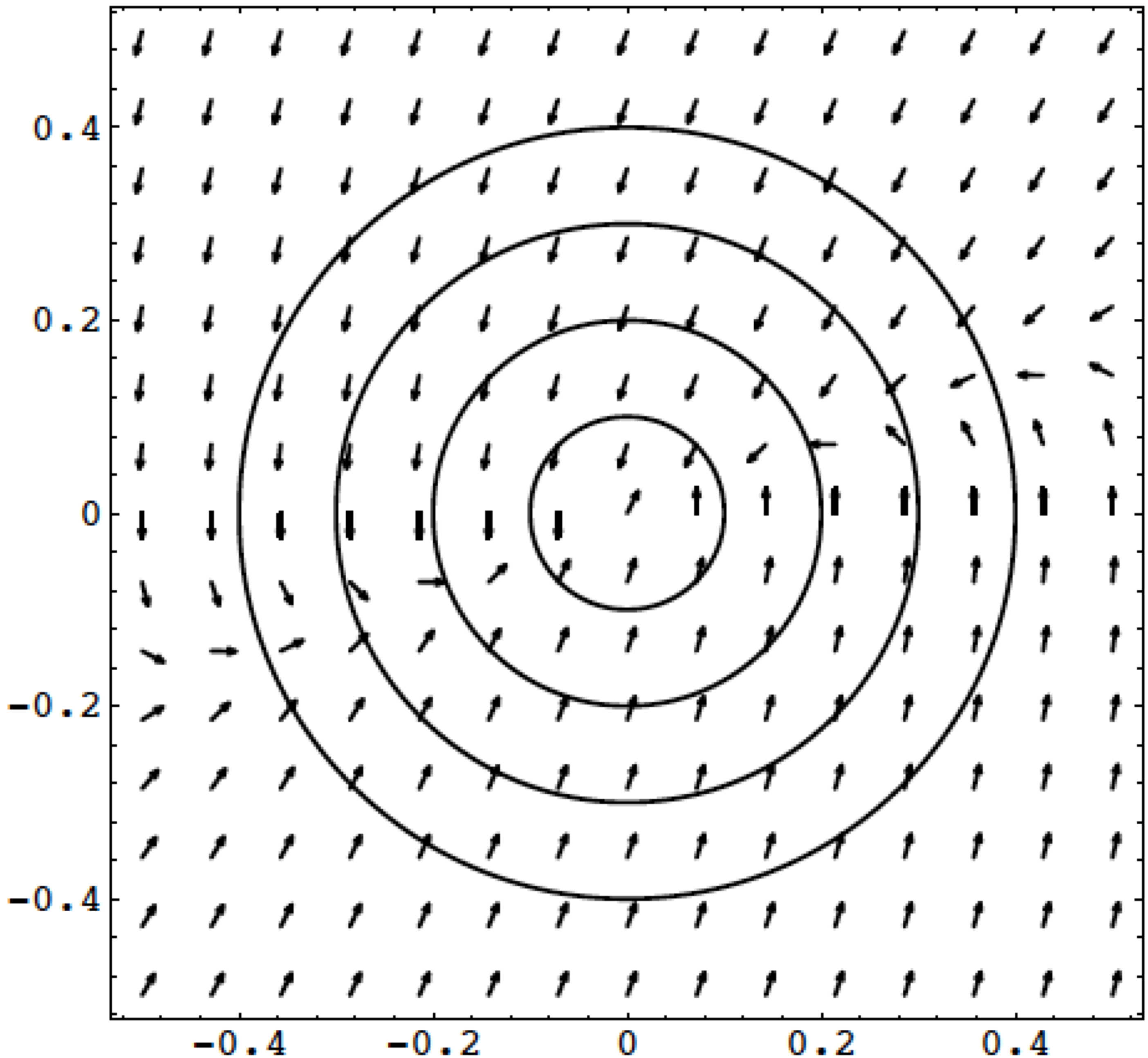
$$V(\mathbf{x}) = x_1^2 + x_2^2$$

$$\begin{aligned}\frac{dV}{dt} &= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} \\ &= 2x_1\frac{dx_1}{dt} + 2x_2\frac{dx_2}{dt} \\ &= 2x_1(-x_2) + 2x_2(x_1 + x_2^3 - 3x_2) \\ &= -2x_1x_2 + 2x_2x_1 + 2x_2^4 - 6x_2^2 \\ &= 2x_2^4 - 6x_2^2 \\ &= 2x_2^2\left(x_2^2 - 3\right).\end{aligned}$$

$-\sqrt{3} < x_2 < \sqrt{3}$ $\quad dV/dt < 0$ → The origin is a stable fixed point



2
1
0
−1
−2
−2
−1
0
1
2



0.4
0.2
0
−0.2
−0.4
−0.4
−0.2
0
0.2
0.4

E.g.4 梯度系统Gradient system

Let $h: \mathbf{R}^n \to \mathbf{R}$, i.e., h is a function of n variables which returns a single number answer. The *gradient* of h, denoted by ∇h, is the vector of h's partial derivatives.

$$h(\mathbf{x}) = h\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_1^2 + x_2^2)^2 = x_1^4 + 2x_1^2x_2^2 + x_2^4,$$

$$\nabla h = \begin{bmatrix} \partial h/\partial x_1 \\ \partial h/\partial x_2 \end{bmatrix} = \begin{bmatrix} 4x_1^3 + 4x_1x_2^2 \\ 4x_1^2x_2 + 4x_2^2 \end{bmatrix}$$

$$\dot{x} = -\nabla h(x) = \begin{bmatrix} -4x_1^3 - 4x_1x_2^2 \\ -4x_1^2x_2 - 4x_2^2 \end{bmatrix}$$

h(X)可以作为李亚普洛夫函数：

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$= \nabla h(\mathbf{x}) \cdot [-\nabla h(\mathbf{x})] = -[\nabla h(\mathbf{x}) \cdot \nabla h(\mathbf{x})] = -|\nabla h(\mathbf{x})|^2$$

考虑系统 $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1e^{x_2} \\ -x_1^2e^{x_2} - 2x_2 \end{bmatrix}$

$$f_1(\mathbf{x}) = -2x_1e^{x_2} \qquad f_2(\mathbf{x}) = -x_1^2e^{x_2} - 2x_2$$

$$\frac{\partial f_1}{\partial x_2} = -2x_1e^{x_2} = \frac{\partial f_2}{\partial x_1}$$

➡ 是梯度系统

求势函数：

$$h(\mathbf{x}) = \int -f_1(\mathbf{x})\,dx_1 = \int 2x_1e^{x_2}\,dx_1 = x_1^2e^{x_2} + C(x_2)$$

$$\frac{\partial h}{\partial x_2} = -f_2 = x_1^2e^{x_2} + 2x_2$$ 同时有 $$\frac{\partial h}{\partial x_2} = x_1^2e^{x_2} + C'(x_2)$$

$$C'(x_2) = 2x_2$$

➡

$$h(\mathbf{x}) = x_1^2e^{x_2} + x_2^2$$

E.g. 5

对动力系统 $\dot{x} = f(x)$，构造函数满足：

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = \nabla V \cdot f(x) \le 0$$

取 $g = \nabla V$ ，如果 $g \cdot f \le 0$, $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$ ，则稳定。

$$\dot{x} = y$$
$$\dot{y} = -kx^3 - y$$

$$g_1 = (1 + 2kx^2)x + y \quad g_2 = x + 2y \qquad \frac{\partial g_1}{\partial y} = 1 = \frac{\partial g_2}{\partial x}$$

$$g_1 f_1 + g_2 f_2 = [(1 + 2kx^2)x + y]y + [x + 2y][-kx^3 - y] = -[kx^4 + y^2] \le 0$$

通过对g积分可以求出V为：

$$V = y^2 + xy + \frac{1}{2}x^2 + \frac{1}{2}x^4 = (y + \frac{1}{2}x)^2 + \frac{1}{4}x^2 + \frac{1}{2}x^4 \ge 0$$

线性稳定性分析

基本想法是如果系统离开某一状态微小距离后，考查该系统是否随着时间远离该状态。若不远离就是稳定，否则为不稳定。不仅可以用于定态，也可以用于随时间演化的状态。

对于连续动力系统:

$$\varphi^t(\bar{x}+x)=\varphi^t(\bar{x})+e^{Df(\bar{x})t}x$$

若$Df(\bar{x})$ 的特征根的实部全部小于零，则$\varphi^t(\bar{x}+x)$会靠近$\varphi^t(\bar{x})=\bar{x}$，从而$\bar{x}$为渐近稳定

对于**一维动力系统:** $\dot{x}=f(x)$

令 $f(\bar{x})=0$可求得定态解$\bar{x}$，$Df(\bar{x})=\left.\frac{df}{dx}\right|_{\bar{x}}$

当 $\left.\frac{df}{dx}\right|_{\bar{x}}<0$时，$\bar{x}$ 稳定;

当 $\left.\frac{df}{dx}\right|_{\bar{x}}>0$时，$\bar{x}$ 不稳定。

对于**二维动力系统:**

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

令$\begin{cases} f(\bar{x}, \bar{y}) = 0 \\ g(\bar{x}, \bar{y}) = 0 \end{cases}$ 可求得其定态解，$Df(\bar{x}) = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(\bar{x},\bar{y})} = J_{(\bar{x},\bar{y})}$

当$J_{(\bar{x},\bar{y})}$的特征根实部全部不大于零，$(\bar{x}, \bar{y})$稳定;

当$J_{(\bar{x},\bar{y})}$的某一个特征根实部大于零，$(\bar{x}, \bar{y})$不稳定。

离散的动力系统:

$x_{t+1} = f(x_t)$，其定态为$\bar{x} = f(\bar{x})$，若对$\bar{x}$ 附近$\bar{x} + x$ 出发的轨道，

$$\bar{x} + x_{t+1} = f(\bar{x} + x_t) = f(\bar{x}) + Df(\bar{x})x_t$$

$$x_{t+1} = Df(\bar{x})x_t \quad \Rightarrow \quad x_t = [Df(\bar{x})]^t x_0$$

因此，如果$\bar{x}$稳定，则要求$Df(\bar{x})$的特征根的模$\|\lambda\| \leqslant 1$

E.g.

① $\dot{x} = x - x^3$

定态 $x - x^3 = 0 \Rightarrow \bar{x}_1 = 0,\quad \bar{x}_{2,3} = \pm 1$

$\bar{x}_1$： $\dot{x} = \frac{d}{dx}(x - x^3)\,|_{x=0}\, x = x$,即 $\dot{x} = x$

所以$\bar{x}_1$是不稳定的平衡点。

$\bar{x}_{2,3}$： $\dot{x} = \frac{d}{dx}(x - x^3)\,|_{x=\pm 1} = -2x$,即 $\dot{x} = -2x$

所以$\bar{x}_{2,3}$是稳定的平衡点。

② $\begin{cases} \dot{x} = x(3-x-2y) \\ \dot{y} = y(2-x-y) \end{cases}$，即 $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$

定态为: $\begin{cases} \bar{x}_1 = 0 \\ \bar{y}_1 = 0 \end{cases}$, $\begin{cases} \bar{x}_2 = 0 \\ \bar{y}_2 = 2 \end{cases}$, $\begin{cases} \bar{x}_3 = 3 \\ \bar{y}_3 = 0 \end{cases}$, $\begin{cases} \bar{x}_4 = 1 \\ \bar{y}_4 = 1 \end{cases}$

线性近似方程为: $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{(\bar{x},\bar{y})} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}\Bigg|_{(\bar{x},\bar{y})} \begin{pmatrix} x \\ y \end{pmatrix}$

对(0,0) $\lambda = 3,2$ $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(0,0)不稳定，轨线沿着$\lambda = 2$的特征矢量(0,1)方向

对(0, 2) $\lambda = -1, -2$ $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(0, 2)稳定，轨线沿着$\lambda = -1$的特征矢量(1, −2)是接近(0, 2):

对(3, 0) $\lambda = -3, -1$ $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(3, 0)稳定，轨线沿着$\lambda = -1$的特征矢量(3, −1)是接近(3, 0)

对(1, 1) $\lambda = -1 \pm \sqrt{2}$ $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(1, 1) 不稳定，鞍点，$\lambda = -1 - \sqrt{2}$的特征矢量$(1, \pm\sqrt{2})$接近(1, 1)，$\lambda = -1 + \sqrt{2}$的特征矢量$(1, \pm\sqrt{2})$ 远离(1, 1)

③ 离散动力学系统: $x_{t+1} = \alpha x_t(1 - x_t)$

定态 $\bar{x} = \alpha\bar{x}(1 - \bar{x}) \Rightarrow \bar{x} = 1 - \frac{1}{\alpha}$

$$\frac{\partial f}{\partial x} = \alpha - 2\alpha x \,|_{\bar{x}} = \alpha - 2\alpha + 2 = 2 - \alpha$$

当 $|2 - \alpha| > 1$时， $\bar{x}$不稳定;

当 $|2 - \alpha| \leq 1$时， $\bar{x}$稳定。

④ 时间滞后系统：$\frac{dx}{dt} = (1-x)x_{t-\tau}$

系统有两个定态：$\bar{x}_1 = 0, \bar{x}_2 = 1$，令$X = \bar{x} + x$得到

$$\frac{d}{dt}(\bar{x} + x) = (1 - \bar{x} - x)(\bar{x} + x_{t-\tau})$$

$$\frac{dx}{dt} = (1 - \bar{x}x_{t-\tau}) - \bar{x}x - xx_{t-\tau} + (1 - \bar{x})\bar{x}$$

忽略高阶项$xx_{t-\tau}$得到：

$$\frac{dx}{dt} = (1 - \bar{x}x_{t-\tau}) - \bar{x}x$$

令$x = e^{\lambda t}$对定态$\bar{x} = 0$有$\lambda e^{\lambda t} = e^{\lambda(t-\tau)}$，即$\lambda = e^{-\lambda t}$得到$\lambda > 0$所以不稳定。

对于定态$\bar{x} = 1 \rightarrow \lambda = -1$，容易判定该定态是稳定的。

⑤ 考虑线性时滞系统:

$$\frac{dx}{dt} + \alpha x_{t-\tau} + \beta x = 0$$

系统的定态为$\bar{x} = 0$，令$x = e^{\lambda t}$，有$\lambda + \alpha e^{-\lambda\tau} + \beta = 0$，

根据α和β 的取值，可以判定λ的正负，从而确定定态的稳定性

⑥ 考虑一个二维动力系统(反应扩散方程) The Brusselator

The Reactions

$$A \rightarrow X$$
$$B + X \rightarrow Y + D$$
$$2X + Y \rightarrow 3X$$
$$X \rightarrow E$$

$$\dot{x} = 1 - (1+b)x + ax^2 y$$

$$\dot{y} = bx - ax^2 y$$

$$\dot{x} = f(x, y) = 1 - (1+b)x + ax^2 y$$

$$\dot{y} = g(x, y) = bx - ax^2 y$$

<u>Fixed Points</u>

$$f(x^*, y^*) = 0 \qquad 1 - (1+b)x^* + ax^{*2}y^* = 0 \qquad x^* = 1$$

$$g(x^*, y^*) = 0 \qquad bx^* - ax^{*2}y^* = 0 \qquad y^* = \frac{b}{a}$$

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix}_{(x^*, y^*)} = \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

$$\lambda_{\pm} = \frac{1}{2}\left((b-a-1) \pm \sqrt{(b-a-1)^2 - 4a}\right)$$

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= f(u,v) = 1-(b+1)u + au^2v + D_1\frac{\partial^2 u}{\partial r^2} \\
\frac{\partial v}{\partial t} &= g(u,v) = bu - au^2v + D_2\frac{\partial^2 v}{\partial r^2}
\end{aligned}
$$

定态$(\bar{u},\bar{v}) = (1, -\frac{b}{a})$，取 微扰项$u_1 \equiv \Delta u = u - 1$，$v_1 \equiv \Delta v = v - \frac{b}{a}$

$$
\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} &= \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} D_1\frac{\partial^2}{\partial r^2} & 0 \\ 0 & D_2\frac{\partial^2}{\partial r^2} \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\
&= \begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} D_1\frac{\partial^2}{\partial r^2} & 0 \\ 0 & D_2\frac{\partial^2}{\partial r^2} \end{pmatrix}\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}
\end{aligned}
$$

令 $$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = e^{ikr} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

得

$$\frac{d}{dt}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} b-1-D_1k^2 & a \\ -b & -a-D_2k^2 \end{pmatrix}\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$$

得到特征根方程:

$$\lambda^2 + \lambda(a-b+1+(D_1+D_2)k^2) + a + aD_1k^2 - (b-1)D_2k^2 + D_1D_2k^4 = 0$$

Routh-Hurwitz 判据

$$|\lambda I - A| = a_n + a_{n-1}\lambda + a_{n-2}\lambda^2 + \cdots + a_1\lambda^{n-1} + a_0\lambda^n = 0$$

$$H = \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & \cdots & \cdots & 0 & 0 \\ \cdots & a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ \cdots & a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ & & \cdots & \cdots & a_{n-3} & a_{n-1} & 0 \\ & & \cdots & \cdots & a_{n-4} & a_{n-2} & a_n \end{pmatrix}$$

根据 Routh-Hurwitz 判据, 该定态稳定的充分必要条件是特征方程的所有系数大于零 ($a_i > 0$) 且 Routh-Hurwit 矩阵的行列式和各阶顺序主子式 (leading principal minor) 均大于零, 即 $\Delta_1 = a_1 > 0, \Delta_2 = a_1a_2 - a_0a_3 > 0, \cdots, \Delta_k > 0 (k = 1, 2, \cdots, n-1)$, 其中第 k 阶顺序主子式 Δ_k 是 H 前 k 列 k 行的块矩阵的行列式.