关联函数对空间距离的依赖关系

In a finite system with N spins, the correlation C_{ij} between spin i positioned at $\boldsymbol{r}_i$ and spin j positioned at $\boldsymbol{r}_j$ can be calculated as [13]:

$$C_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle, \tag{5}$$

where the averages are done over all configurations with weight $p(\{S_i\})$. Using C_{ij} as its elements, a $N \times N$ correlation matrix $\boldsymbol{C}$ can be obtained. C has N eigenvectors and eigenvalues. For eigenvector $\boldsymbol{b}_n$ of eigenvalue λ_n, there is the relation

系统的关联函数：

$$\boldsymbol{C}\boldsymbol{b}_n = \lambda_n \boldsymbol{b}_n, \ n = 1, 2, ..., N,$$

where

$$\boldsymbol{b}_n = \begin{bmatrix} b_{1n} \\ b_{2n} \\ \cdot \\ \cdot \\ \cdot \\ b_{Nn} \end{bmatrix}.$$

关联矩阵本征矢量：

The normalized eigenvectors are orthogonal to each other and satisfy the conditions

$$\boldsymbol{b}_n \cdot \boldsymbol{b}_l = \sum_j b_{jn} b_{jl} = \delta_{nl}, \tag{8}$$

where δ_{nl} is the Kronecker delta.

主涨落模式：

From the fluctuations of N spins, we can define N principal fluctuation modes

$$\delta\widetilde{S}_n = \sum_{j=1}^{N} b_{jn}\delta S_j, \ \ n = 1, 2, ..., N. \tag{9}$$

Using the orthogonal conditions in eq. (8), the correlation between principal fluctuation modes can be calculated as:

$$\widetilde{C}_{nl} \equiv \left\langle \delta\widetilde{S}_n \ \delta\widetilde{S}_l \right\rangle = \lambda_n \delta_{nl}. \tag{10}$$

There is no correlation between different principal fluctuation modes. The mean square of principal fluctuation mode $\delta\widetilde{S}_n$ is equal to λ_n.

总关联来自各本征涨落模式的之和：

We can express the correlation C_{ij} between spin i and spin j by eigenvectors and eigenvalues as:

$$C_{ij} = \sum_{n=1}^{N} b_{in} b_{jn} \lambda_n \ . \tag{11}$$

We introduce $C_{ij}^{(n)} = b_{in} b_{jn}$, which can be understood as the correlation between spin i and spin j in the n-th principal fluctuation mode. The total correlation between spin i and spin j is obtained by summing correlations of all principal fluctuation modes as:

$$C_{ij} = \sum_{n=1}^{N} \lambda_n C_{ij}^{(n)}. \tag{12}$$

本征涨落模式的权重：

In ref. [13], it has been proposed and confirmed by the two-diemnsional Ising model that eigenvalues satisfy the following finite-size scaling form

$$\lambda_n(t, L) = L^{2-\eta} f_n(tL^{1/\nu}), \tag{13}$$

where η is related to the critical exponent γ of susceptibility by the hypescaling relation $2 - \eta = \gamma/\nu$. Using position vectors $\boldsymbol{r}_i$ of spins, eigenvector $\boldsymbol{b}_n$ can represented by a spatial function $b_n(\boldsymbol{r}_i)$. In a finite system of volume L^d and periodic

本征矢量的空间分布：

boundary conditions, the spatial function can be expressed as:

$$b_n(\boldsymbol{r}_i) = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{k}} \hat{b}_n(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r}_i) \tag{14}$$

with the Fourier component

$$\hat{b}_n(\boldsymbol{k}) = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{r}_i} b_n(\boldsymbol{r}_i) \exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}_i), \tag{15}$$

where $\boldsymbol{k}$ has components $k_j = 2\pi m_j/L$, $m_j = 0, \pm 1, \pm 2, \ldots,$ $j = 1, 2, \ldots, d$ in the range $-\pi \leq k_j < \pi$. Fourier components follow the relation $\hat{b}_n^*(\boldsymbol{k}) = \hat{b}_n(-\boldsymbol{k})$ according to eq. (15). The

关联函数的标度形式：

follow the relation $\hat{b}_n^*(\boldsymbol{k}) = \hat{b}_n(-\boldsymbol{k})$ according to eq. (15). The correlation function of distance can be calculated as:

$$g(\boldsymbol{r}, t, L) = \frac{1}{N} \sum_{\boldsymbol{r}_i} \sum_{n=1}^{N} b_n(\boldsymbol{r}_i) b_n(\boldsymbol{r}_i + \boldsymbol{r}) \lambda_n. \tag{16}$$

By using the orthogonal condition

$$\frac{1}{N} \sum_{\boldsymbol{r}_i} \exp(\mathrm{i}\boldsymbol{k}\boldsymbol{r}_i) \exp(\mathrm{i}\boldsymbol{k}'\boldsymbol{r}_i) = \delta_{\boldsymbol{k}+\boldsymbol{k}',0} \tag{17}$$

and the finite-size scaling form of eigenvalues in eq. (13), the correlation function of distance can be expressed as:

$$g(\boldsymbol{r}, t, L) = L^{-d+2-\eta} \sum_{n} \sum_{\boldsymbol{k}} f_n(tL^{1/\nu}) \left|\hat{b}_n(\boldsymbol{k})\right|^2 \exp(-\mathrm{i}\boldsymbol{k}\boldsymbol{r}), \tag{18}$$

where $\boldsymbol{k} = 2\pi\mathbf{m}/L$.

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From this result, we propose that the correlation function of distance follows the finte-size scaling form

$$g(\boldsymbol{r}, t, L) = A\ |\boldsymbol{r}|^{-d+2-\eta}\, G\left(\boldsymbol{r}/L, tL^{1/\nu}\right), \quad (19)$$

for $|t| \ll 1$ and $L \gg \tilde{a}$. It is expected that the directional de-

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for $|t| \ll 1$ and $L \gg \tilde{a}$. It is expected that the directional dependence of the universal scaling function $G\left(\boldsymbol{r}/L, tL^{1/\nu}\right)$ can be negleted for $|\boldsymbol{r}| \ll L$ and becomes unnegligible when $|\boldsymbol{r}|$ is comparable to L.

系统无限大：

Correlation functions describe how microscopic variables at different positions are related. In an infinite system with a lattice spacing $\tilde{a}$, the correlation function $g(\boldsymbol{r}, t, \infty)$ reads [1, 2] for $|t| \ll 1$ and $|\boldsymbol{r}| \gg \tilde{a}$,

$$g(\boldsymbol{r}, t, \infty) = A\, |\boldsymbol{r}|^{-d+2-\eta} \Phi(|\boldsymbol{r}|/\xi) \qquad (4)$$

with a universal scaling function Φ, a nonuniversal amplitude A, and with $\xi = \xi_0 |t|^{-\nu}$, apart from correlations to scaling.

$$\boxed{G\left(\boldsymbol{r}/L, tL^{1/\nu}\right)} \quad \underset{L \to \infty}{\Rightarrow} \quad \boxed{\Phi(|\boldsymbol{r}|/\xi)}$$

关联函数比值不动点对应临界点：

Using the correlation function at distances $x, 2x, 4x$ along the x-direction in the lattice with sizes $L, 2L, 4L$ respectively, we can define a ratio

$$R = \frac{g(x,t,L)g(4x,t,4L)}{g(2x,t,2L)^2}. \tag{23}$$

This ratio can be written into a scaling form as:

$$R(\lambda, tL^{1/\nu}) = \frac{G(\lambda, tL^{1/\nu})G(\lambda, 2^{2/\nu}tL^{1/\nu})}{G(\lambda, 2^{1/\nu}tL^{1/\nu})^2}, \tag{24}$$

where $\lambda = x/L$. At critical point $t = 0$, we have $R(\lambda, 0) = 1$, which is independent of λ and L. At $t \neq 0$, the ratio R depends on both λ and L. We can determine the critical point of a system from the fixed point of R as a function of T for different λ and L.

关联函数标度形式的蒙特卡洛模拟验证

To verify the finite-size scaling structure of correlation function in eq. (19), we investigate correlation functions of the Ising model and the bond percolation in two-dimensional lattice with PBC using the Monte Carlo simulation. The lattice sizes $L = 32$, 64, and 128 are taken in our simulations.

3 Correlation function of Ising model

At zero external field, the Ising model has the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \tag{25}$$

伊辛模型的蒙特卡洛模拟

At zero external field, the Ising model has the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i S_j, \tag{25}$$

where interactions are restricted to the nearest neighbors and spins can take two values, $S_i \ni \{-1, +1\}$. A configuration of system is characterized by $\{S_i\} = (S_1, S_2, ..., S_N)$ with $N = L \times L$. It has a probability $p(\{S_i\}) = \mathrm{e}^{-H/k_\mathrm{B}T}/Z$ with $Z = \sum_{\{S_i\}} \mathrm{e}^{-H/k_\mathrm{B}T}$, where the summation is done for all samples of configuration. The Wolff algorithm [14] is used to simulate configurations of the Ising model.

距离关联函数的获得

For N spins in the lattice, there are $N(N_1)/2$ correlations between spins. The correlation function $g(\boldsymbol{r}, t, L)$ is calculated by the average of correlations C_{ij} with $\boldsymbol{r}_i - \boldsymbol{r}_j = \boldsymbol{r}$ as:

$$g(\boldsymbol{r}, t, L) = \frac{\sum_{i,j} C_{ij} \delta\left[\boldsymbol{r} - (\boldsymbol{r}_i - \boldsymbol{r}_j)\right]}{\sum_{i,j} \delta\left[\boldsymbol{r} - (\boldsymbol{r}_i - \boldsymbol{r}_j)\right]}. \quad (26)$$

二维伊辛的临界点及临界指数

It has been found that the two-dimensional Ising model in square lattice has the critical point at $k_B T_c/J = 2/\ln(1+\sqrt{2}) \approx 2.269$ [15], and the critical exponents $\nu = 1$ and $\eta = \frac{1}{4}$. To verify the finite-size scaling structure of correlation function, we simulate the Ising model of sizes $L = 32, 62, 128$ around the critical point with $tL^{1/\nu} = -2, 0, 2$.

临界点关联函数的标度性

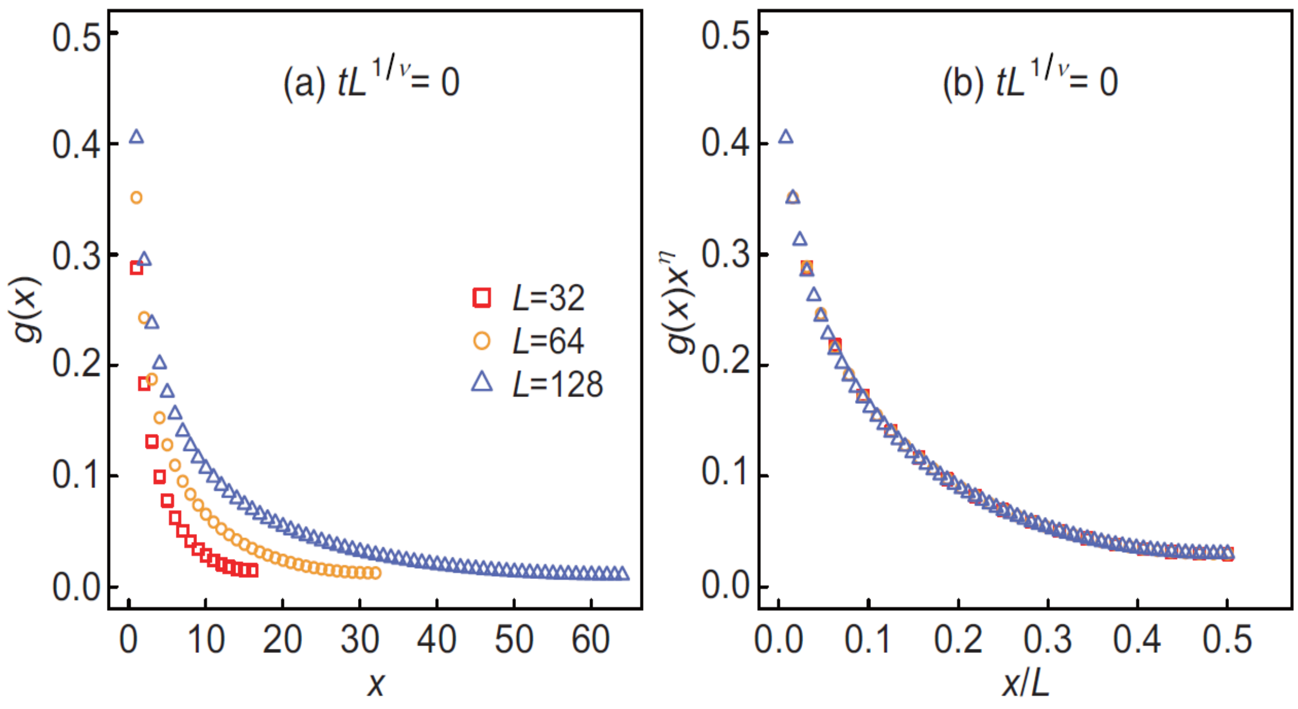

Figure 1 (Color online) (a) Correlation function along x-axis of the Ising model at T_c; (b) its finite-size scaling function.

临界温度一下关联函数的标度性

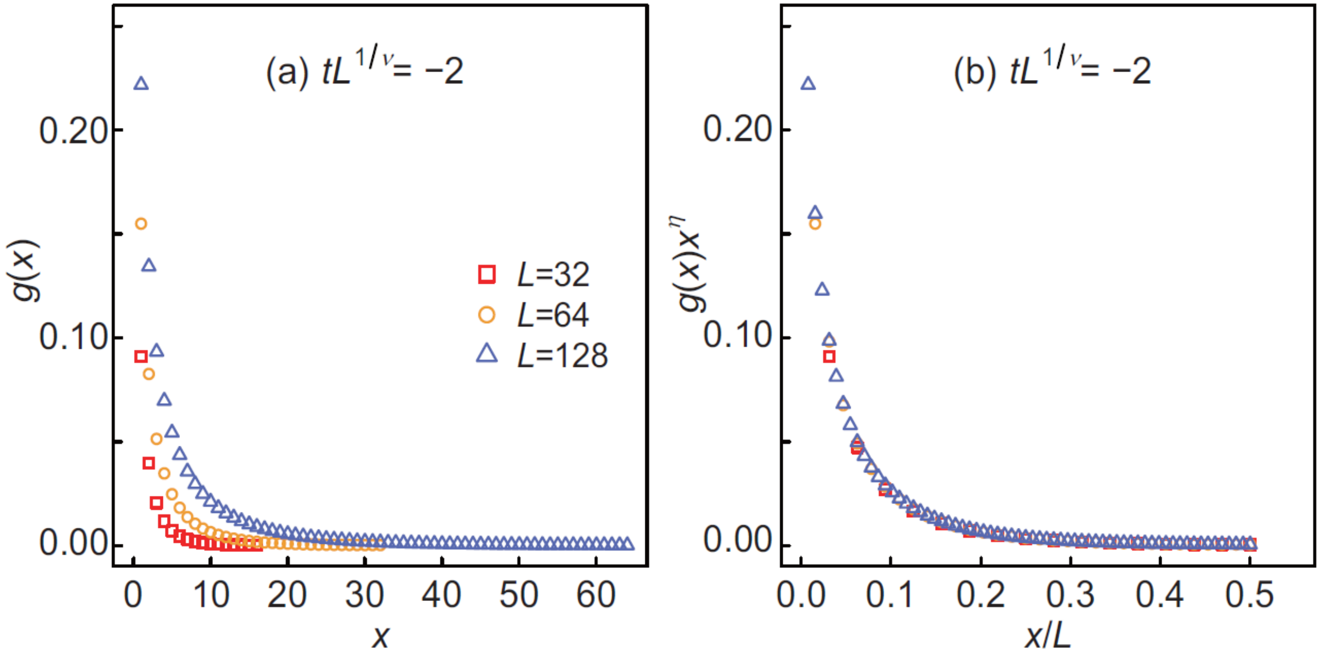

Figure 2 (Color online) (a) Correlation function along x-axis of the Ising model below T_c; (b) its finite-size scaling function.

临界温度以上关联函数的标度性

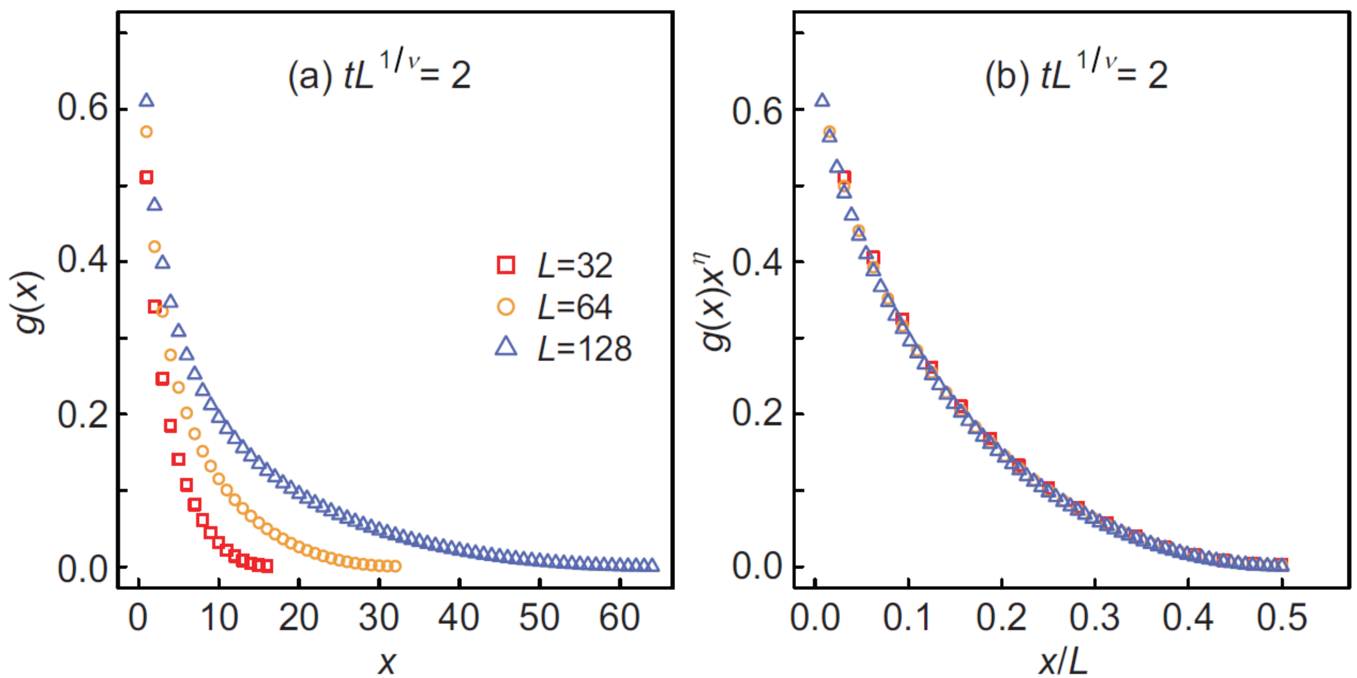

Figure 3 (Color online) (a) Correlation function along x-axis of the Ising model above T_c; (b) its finite-size scaling function.

对角线与轴线方向关联函数比较：

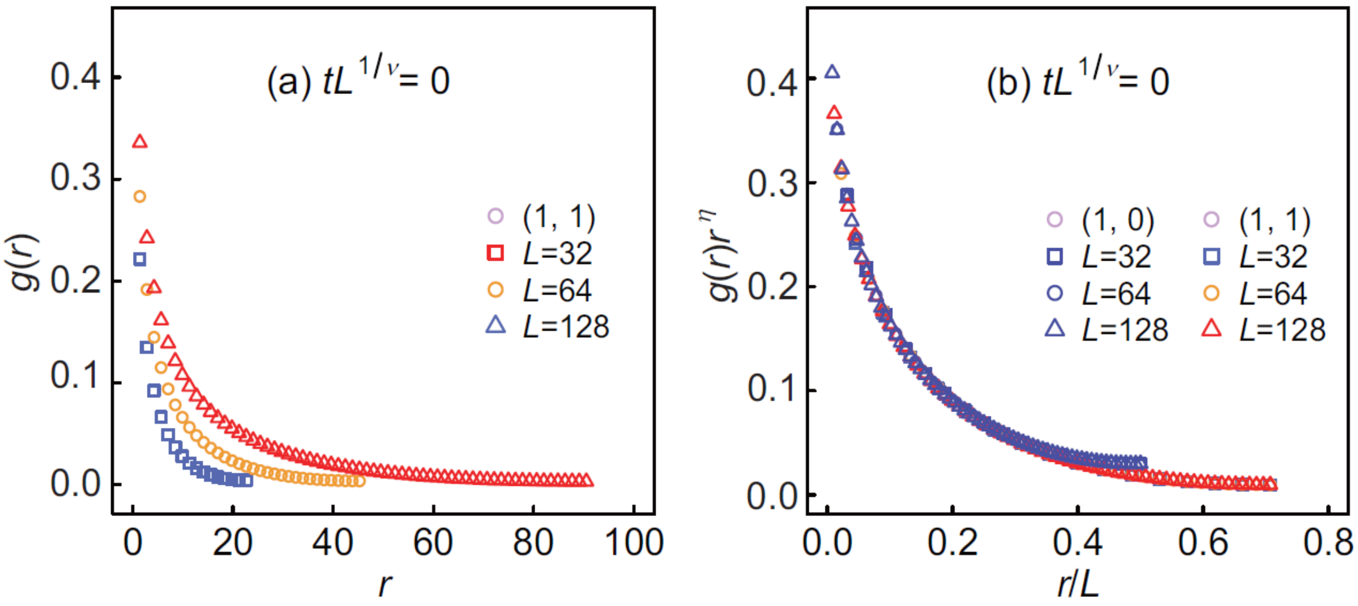

Figure 4 (Color online) (a) Correlation function along diagonal direction of the Ising model at critical point; (b) its finite-size scaling function in comparison with that along *x*-axis.

对角线与轴线方向关联函数的比较

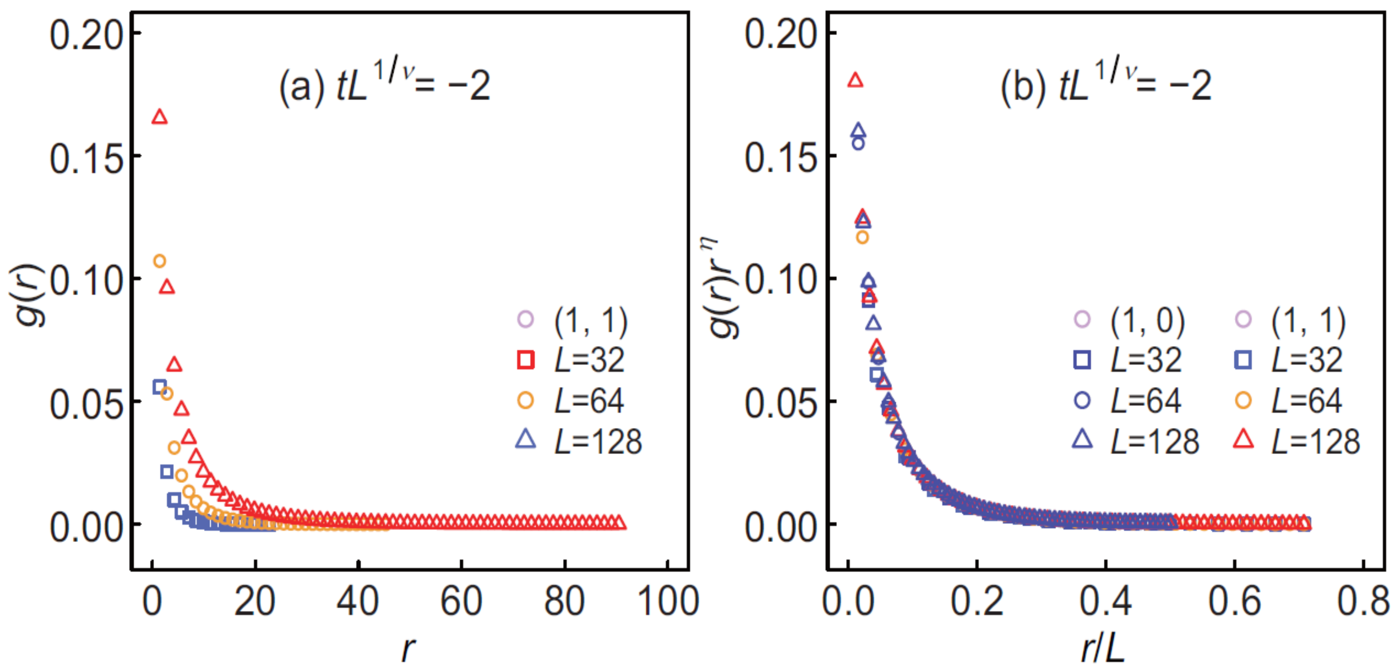

Figure 5 (Color online) (a) Correlation function along diagonal direction of the Ising model below T_c; (b) its finite-size scaling function in comparison with that along x-axis.

对角线与轴线方向关联函数的比较：

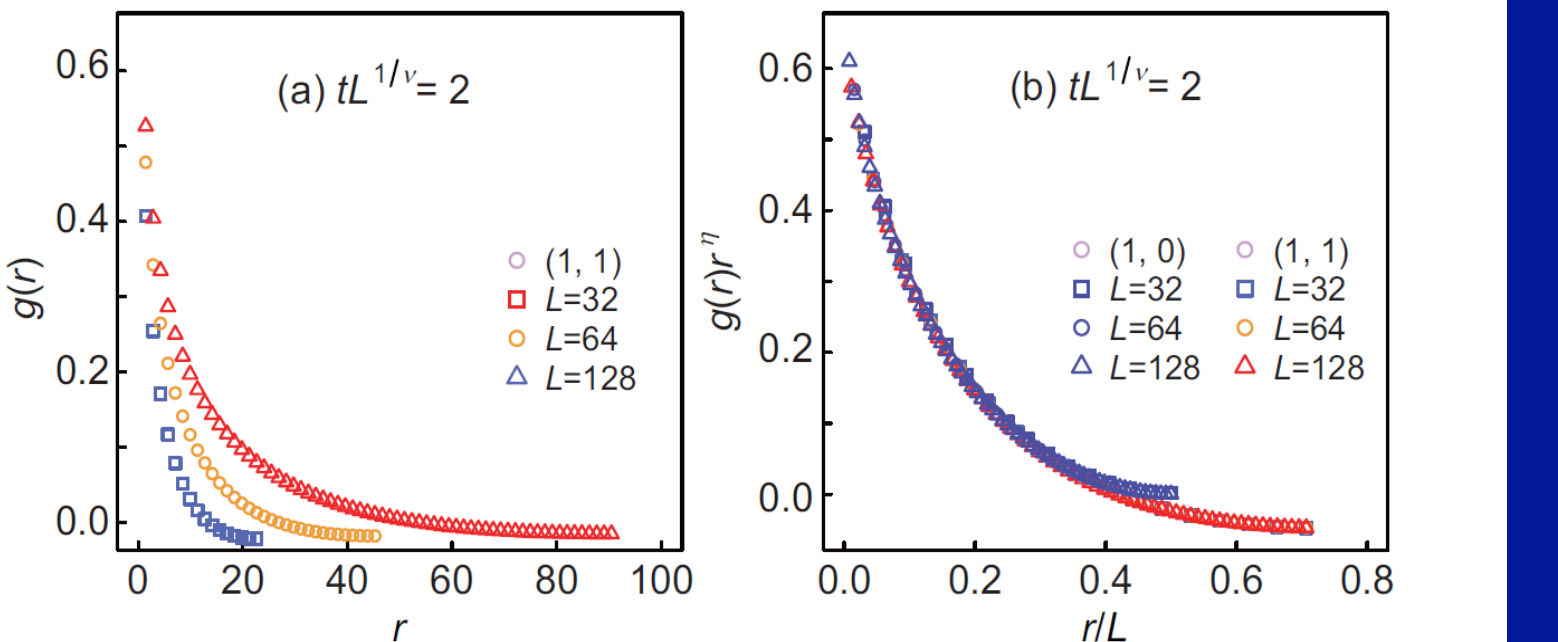

Figure 6 (Color online) (a) Correlation function along diagonal direction of the Ising model above T_c; (b) its finite-size scaling function in comparison with that along x-axis.

临界点t=0，r/L固定，关联函数的标度性

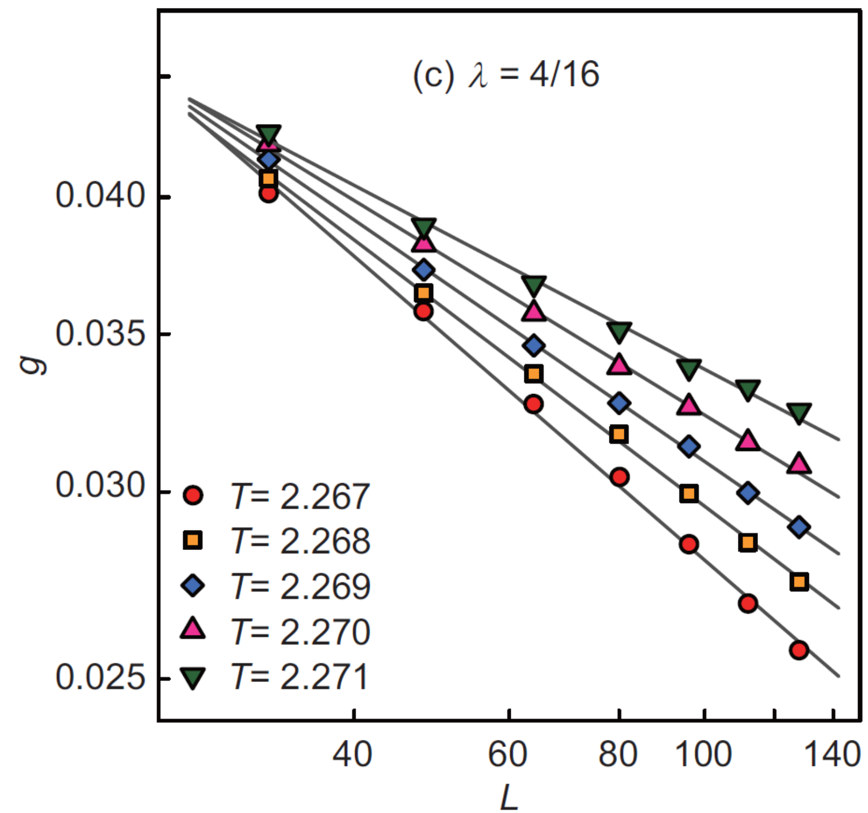

Figure 7 (Color online) Log-log plot of correlation function of the Ising model along x-axis $g(\lambda L, t, L)$ with respect to system size L.

$$g(\boldsymbol{r}, t, L) = A \ |\boldsymbol{r}|^{-d+2-\eta} G\left(\boldsymbol{r}/L, tL^{1/\nu}\right),$$

关联函数比值的不动点：

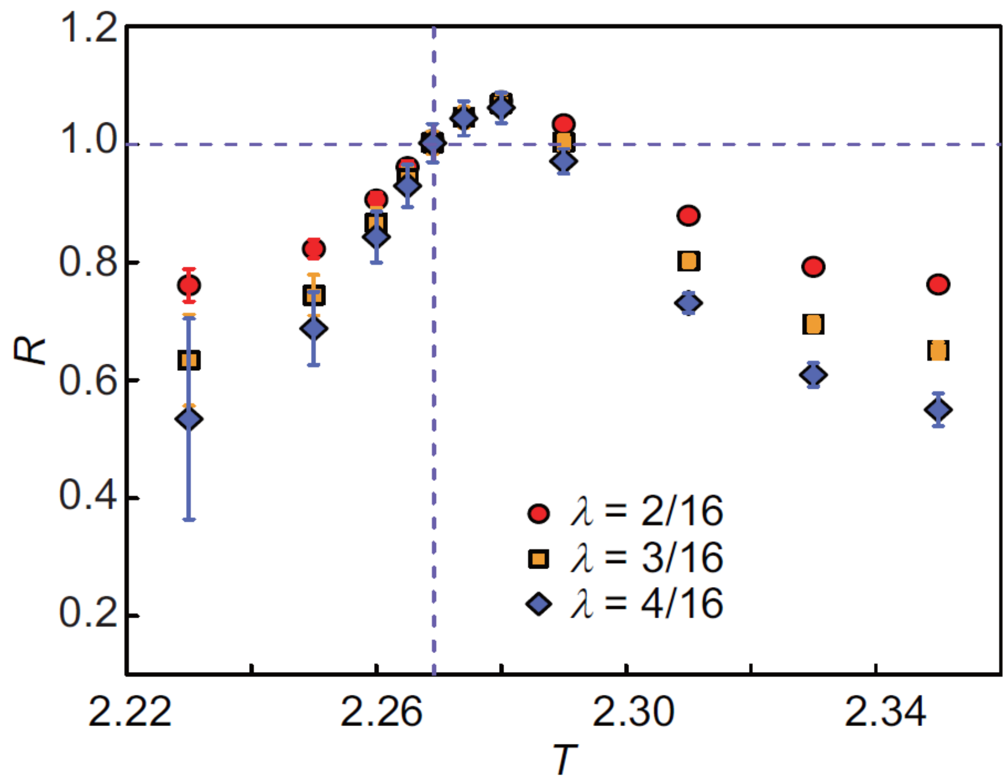

Figure 8 (Color online) Ratio $R(\lambda, tL^{1/\nu})$ of the Ising model with respect to temperature at L = 32 and different λ.

结论：

Here, we have proposed the finite-size scaling of correlation functions in finite systems near their critical points. For a finite system of size L at the reduced temperature $t = (T - T_c)/T_c$, the correlation function at a distance $\boldsymbol{r}$ can be scaled as $g(\boldsymbol{r}, t, L) = A|\boldsymbol{r}|^{-(d-2+\eta)}G(\boldsymbol{r}/L, tL^{1/\nu})$. At distances $|\boldsymbol{r}| \ll L$, the finite-size scaling function $G(\boldsymbol{r}/L, tL^{1/\nu})$ is independent of the direction of $\boldsymbol{r}$, but it has a significant directional dependence when the distance is compariable to L. From the finite-size scaling of the correlation function, we can obtain the finite-size scalings of the susceptibility [1] and the second moment correlation length [2].